# VALUATION OF RISKY DEBT:
# A MULTI-PERIOD BAYESIAN FRAMEWORK.

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**Abstract.**

The paper describes model of a new type for valuation of risky bonds and loans that we call Bayesian Multi-Period (BMP) model. BMP is neither structural nor reduced form model and not a Merton-type model at all. It considers exact contractual cash flow schedule between lender and borrower and combines it with detailed multi-period prognosis of borrower's default at all stages cash flow process. The prognosis can be based on indices of borrower's current financial position, market factors like "distance to default" or credit spreads as well as on newly proposed indices like a schedule of future paying off a firm's long-term debt. As a result the model calculates "fair" value of a risky debt, "fair" risky yield to maturity, "fair" interest rate on risky bond (at time of issuance), credit loss distribution etc.

The model explains on average about 70% of observed credit spreads.

Keywords: Credit risk, Risky bonds, Default prediction, Bond valuation, Bayesian model.

GEL codes: C11, C13, C51, E58, G21, G33.

## Introduction

Problem of measuring of credit risk is one of the central problems of modern finance. Its correct and constructive decision is important for banks, lending money, firms issuing bonds, borrowers, investors, regulators that control banking activities. The problem is interesting and important for numerous scientists working in the area of finance.

In recent times interest to the problem became greatly stimulated by developing of the New Basel Capital Accord (Basel II) that establishes the new legal standards for measurement methodology and value of adequate bank capital with strong emphasis on improvement of bank capabilities to assess and manage credit (and other) risks.

Modern financial theory of credit risk is dominated by structural approach that goes back to Black, Scholes (1973) diffusion model of dynamics of market value assets of a firm developed for pricing of options and its extension on pricing of risky debt proposed by Merton (1974). In these models default (bankruptcy) of a firm on its obligations (driving factor of credit risk) occurs if market value assets in its random motion fall behind a book value of a firm's debt, consisting of a single type zero coupon bonds. The approach was very fruitful in the past times and created ground for many past and present financial studies. Among these studies one can find a number of extensions of Black-Scholes-Merton model (BSM) aimed on reduction of various restrictive assumptions used in original BSM.

Among practical applications of structural approach most familiar is Moody's KMV EDF (Expected Default Frequency) model that calculates probability (frequency) of a firm's default at one through five years time horizons. To do this EDF calculates the only predictive variable - distance to default (DD) at a horizon of prediction of default probability and then links DD with EDF using empirical database (see Crosbie, Bohn, 2003). EDFs calculated for longer (2,3,4,5 – year) time horizons are then divided in equal amounts to obtain yearly EDFs[1].

Basel II credit risk model can be viewed as a derivative of structural model. Firm's assets are considered at initial time moment and one year later. Normalized return on assets is viewed as a risk factor (with standard normal distribution). Default threshold for the factor is mapped to a borrower's one-year default probability (as it is assessed by a bank and its supervisor). All other default-relevant characteristics of borrowers are omitted.

Other credit risk models for practical applications, known from literature, are developed mainly by major investment banks and rating agencies. Such models of J.P. Morgan (1997), Credit Suisse First Boston (1997) etc are based on various formal indexes of a client's risk.

[1] Such transformation of EDF into multi-period model is incorrect because lower observable values of DD at time of prognosis will cause earlier defaults (higher yearly EDFs in the first years within prediction interval and lower EDFs in later years) while higher DD values will cause later defaults. The difference can be significant.

One can also view Basel II model as a product of joint revaluation of those models.

Notwithstanding many achievements in developing structural models they failed to overcome serious drawbacks.

Employed in the approach economic condition of default (market value assets fall behind book value of a borrower's debt) rarely holds in practice. Empirical data shows that this condition is neither necessary nor sufficient. In our sample (see Philosophov, Batten, Philosophov (2003)) we have a number of going concerns that have negative net worth for several years while other filed for bankruptcy with positive net worth.

Corporate debt is usually diversified and consists of many bonds (and loans) with different maturities and terms of issuance. This increases firm's ability to maneuver and thus reduce its debt burden.

Structural models are one-period while credit process in its nature is multi-period – includes many cash flows spread over the time of credit period. Credit loss depends on the stage at which default occurs.

Structural approach has restricted abilities of mapping credit risk model to each specific firm, its financial position, characteristics of its bonds and loans. In particular the approach acknowledges the only prognostic factor – distance to default (DD) – and claims its exclusive and exhaustive character. At the same time many studies (Hillegeist, Cram, Keating, Lundstedt (2002), Bharath, Shumway (2004), Duffie, Saita, Wang (2005)) report finding other predictive variables that provide incremental information to DD. Structural credit risk model can hardly incorporate those additional predictive variables.

As a result many authors (Jones, Mason, Rosenfeld (1984); Kim, Ramaswamy, Sundaresan (1993); Ogden (1987); Lyden, Saraniti (2000); Leland (2002); Eom, Helwege and Huang (2004)) report that structural models are imprecise and mainly undervalue credit risk.

According to Bohn (2000), "The conventional wisdom, while praising the theoretical insights gained from structural models, dismisses them as impractical for actual bond valuation".

To overcome above described drawbacks we propose an alternative - Bayesian Multi-period (BMP) model that has much more clear perspectives of practical applications.

Its principal features and distinctions are as follows:

- BMP is discrete time model that considers credit process on discrete time intervals matched with actual cash flow schedule of a bond (or loan);
- BMP assesses credit loss for each bond or loan of a firm separately, considering at the same time total debt of a firm while assessing its propensity to default (bankruptcy);

- BMP is multi-period model. It determines in coherent and non-contradictory way probabilities of a firm's default at all future time intervals within active credit period and after its end;
- in contrast to BSM model BMP considers value of a risky bond to be random (function of random event and time of default). This makes applicable such probabilistic concepts as mean value, value variance, marginal value etc;
- BMP does not predict future values of a firm's assets, equity or predictive variables. It uses current (observed) values of predictive variables to calculate probabilities of default at various time intervals in future. After all only current values of predictive variables are observable and their predicted values can be useful if uncertainty of prediction is small. Being calculated within insufficiently substantiated assumptions, predicted values of prognostic variables can bring additional errors.
- The model can use all known default-informative characteristics of a borrower (including distance-to-default) and employs enhanced Bayesian methodology to account for their non-normality and mutual dependence. It provides detailed mapping of model's characteristics to each specific borrower and bond.
- BMP does not make maximum likelihood assessments of conditional hazard rates (ex-post default probabilities) with various assumptions about their dynamics and dependence on predictive variables. It calculates those probabilities using exact probabilistic formulae within enhanced Bayesian methodology. Empirical data is used at intermediate stages to assess conditional distributions of predictive variables by means of widely known and well-established kernel and histogram methods. This principal distinction of BMP model motivates us to call it "Bayesian".

The BMP approach is based on studies L. Philosophov, V. Philosophov (2002) and L. Philosophov, J. Batten, V. Philosophov (2003), which provide multi-period bankruptcy prediction basing on borrower's (a firm's) current financial indices and schedule of repaying of long-term debt. To ensure possibility of independent reading of the current paper we shortly describe below main issues of those studies.

The rest of the paper is organized as follows:

Section 2 represents mathematical inference of BMP model for valuation of risky debt. The model combines a true cash flow schedule of a risky debt with a multi-period default prediction of a debtor at all stages of cash flow process. As a result the model calculates probabilistic distribution of a bond's NPV, its statistical characteristics, taking in account possible recoveries.

Section 3 represents short description of approach and principal issues in multi-period Bayesian default prediction that constitutes principal part of bond valuation model.

Section 4, basing on BMP model, calculates fair interest rates on a risky bond (at time of bond issuance), fair prices and fair yields on a bond at various times through the bond life cycle. Section compares assessed risky prices, yields and interest rates with true interest rates, market prices and yields of real corporate bonds. To make such comparison we simultaneously use accounting data of a firm, information concerning issues of a firm's bonds, market data on bond trades.

Section 5 concludes.

## 2. Bayesian multi-period model for valuation risky bonds.

Suppose that at time $t_0$ a firm issues a bond whose par value is $U$ dollars. A bond pays interest at annual rate $r_b$ at time moments $t_m (m = 1,...M)$ and is redeemed after M years at time $t_M$ for the same U dollars.

Alternatively one can consider a bank that at time moment $t_0$ lends to a client (firm) $U$ dollars. The loan must be returned after $M$ years at time moment $t_M$ and additionally client pays annual interest at rate $r_b$ (at time moments $t_m$).

Consider first the value of a bond (loan) at time of issuance.

If the bond is risk-free its Net Present Value at time $t_0$ can be calculated as

$$V = \sum_{m=0}^{M} \frac{r_b U}{(1 + r_f)^m} + \frac{U}{(1 + r_f)^M}, \qquad (1)$$

where $r_f$ is risk-free discount rate at time $t_0$.

If at time of bond issuance its emitter chooses $r_b = r_f$ he obtains $V = U$.

A firm that issues a bond is subject to default (bankruptcy), which can occur at a random time moment $t_D$. This time moment can lie within one of the following time intervals:

- time interval $T_1 \Rightarrow \{t_0 \div t_1\}$ with probability $P_D(T_1)$;
- time interval $T_2 \Rightarrow \{t_1 \div t_2\}$ with probability $P_D(T_2)$;

…………………………………………………………………..

- time interval $T_M \Rightarrow \{t_{M-1} \div t_M\}$ with probability $P_D(T_M)$;
- time interval $T_{M+} \Rightarrow \{t_M \div t_\infty\}$ with probability $P_D(T_{M+})$.

All intervals except the last one are intervals between adjacent interest payments that can be equal to one year, half of a year etc. Other intervals (in case of more complex cash flow schedules) can be also considered if necessary. The last time interval corresponds to defaults, which occur after the debt is paid off. It includes situation when default does not occur; this situation is referred to as default at infinite time ($t_\infty$).

Note that the described group of default events is full and $P_D(T_1) + .... + P_D(T_M) + P_D(T_{M+}) = 1$.

Methods of calculation of probabilities $P_D(T_m)$ are discussed in section 3.

If a firm defaults before the debt is paid off, cash flows between firm and its lenders cease, and value (NPV) of a bond decreases.

Consider first that no recoveries are possible.

One can see from (1) that value of a bond $V$ can take one of the discrete random values, namely:

- $V = V_1 = 0$ with probability $P_D(T_1)$;
- ……………………………………………………………….
- $V = V_m = \sum_{k=1}^{m-1} \frac{r_b U}{(1+r_f)^k}$ with probability $P_D(T_m)$; (2)
- ……………………………………………………………….
- $V = V_{M+} = \sum_{m=1}^{M} \frac{r_b U}{(1+r_f)^m} + \frac{U}{(1+r_f)^M}$ with probability $P_D(T_{M+})$.

Formulae (2) in combination describe probabilistic distribution of value of a bond. This distribution is of discrete type. Probability density distribution in equivalent continuous form can be written as:

$$P(V) = P_D(T_{M+}) \times \delta(V - V_{M+}) + \sum_{m=1}^{M} P_D(T_m) \times \delta(V - V_m), \tag{3}$$

where $\delta(\bullet)$ is (well known in mathematics) symbolic delta-function, which may be thought of as a normal probability density with very small (almost zero) variance. $\delta(V - V_m)$ is concentrated around $V = V_m$

Cumulative distribution function $F(V)$ of a bond value is a stepwise function. Steps occur at values $V = V_m$ and have magnitudes $P_D(T_m)$:

$$F(V) = P_D(T_{M+}) \times 1(V - V_{M+}) + \sum_{m=1}^{M} P_D(T_m) \times 1(V - V_m), \tag{4}$$

where function $1(x)$ is defined as: $1(x) = 0$ if $x < 0$ and $1(x) = 1$ if $x \geq 0$.

From (3,4) one can easily calculate mean value of a risky bond that is equal to:

$$\overline{V} = V_{M+} \times P_D(T_{M+}) + \sum_{m=1}^{M} V_m \times P_D(T_m) . \qquad (5)$$

Suppose now that after a defaulting firm's recovery procedures are over, bank receives back a portion $\beta$ of lost amount. If $\beta$ is fixed, each random bond value $V_m$ must be increased to

$$V'_m = V_m + \beta(V_{M+} - V_m) = V_{M+}\beta + V_m(1-\beta) .$$

In this case instead of (3) one can obtain probability density of bond value conditional on recovery rate $\beta$ in form:

$$P(V / \beta) = P_D(T_{M+}) \cdot \delta(V - V_{M+}) + \sum_{m=1}^{M} P_D(T_m) \cdot \delta(V - V_{M=} \cdot \beta - V_m \cdot (1-\beta)) , \qquad (6)$$

and mean conditional bond value:

$$\overline{V}(\beta) = V_{M+} \times P_D(T_{M+}) + \sum_{m=1}^{M} (V_{M+}\beta + V_m(1-\beta)) \times P_D(T_m) . \qquad (7)$$

Expression (6) can be further averaged if recovery rate is random:

$$P(V) = \int P(V / \beta) P_r(\beta) d\beta ,$$

where $P_r(\beta)$ is probability density of recovery rate ($P_r(\beta) = 0$ if $\beta \leq 0$ and $\beta \geq 1$), that is sometimes taken as beta-distribution.

The final (unconditional) probability density of a firm's value is:

$$P(V) = P_D(T_{M+}) \cdot \delta(V - V_{M+}) + \sum_{m=1}^{M} P_D(T_m) \cdot \frac{1}{V_{M+} - V_m} \cdot P_r\left(\frac{V - V_m}{V_{M+} - V_m}\right) , \qquad (8)$$

and its cumulative distribution

$$F(V) = P_D(T_{M+}) \cdot 1(V - V_{M+}) + \sum_{m=1}^{M} P_D(T_m) \cdot B\left(\frac{V - V_m}{V_{M+} - V_m}\right) , \qquad (9)$$

where $B(x)$ is cumulative distribution of recovery rate ($\beta$). One can see that probability distribution function $F(V)$ is of combined (discrete plus continuous) type.

A characteristic view of cumulative distributions (3) and (9) in case $r_b = r_f$ is represented in figure 1.

For each value $V$ plot on horizontal axis of the graph ordinate $F(V)$ of cumulative distribution is probability of $NPV$ of the bond to be less than $V$. One can see that in this example probability of $NPV$ be less than $V_{M+} = U$ is nearly 0,1. This is probability of a bond's default within active period of its life.

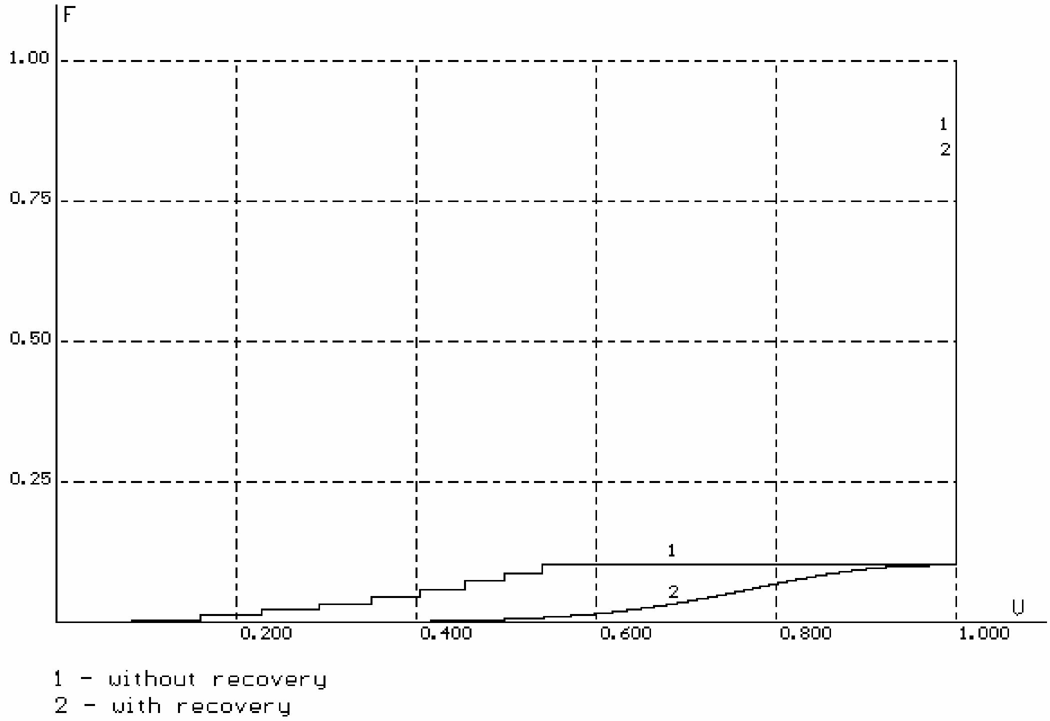


Figure 1. Cumulative distributions of $NPV$ of a risky bond for $r_b = r_f$. Both distributions have big vertical steps at $V=1,0$ whose magnitude is probability that default did not occur within active bond period. The curve has stepwise character if recovery is impossible (curve 1) and combined (continuous plus discrete) character in case of random continuously distributed recovery rate (curve 2).

Generally, if $r_b = r_f$, all possible random values $V_m$, except $V_{M+}$, are less than $NPV$ of risk-free bond with the same parameters (risk-free bond corresponds to $P_D(T_m) = 0$, $(m = 1,..M)$ and $P_D(T_{M+}) = 1$). As a result mean $NPV$ of risky bond will be also less than $NPV$ of equivalent risk-free bond. In the current example $\overline{V} = 93,68\%$ of a bond's face value $U$.

To compensate investors for risky character of bond, issuer must increase interest rate $r_b$ over risk-free rate $r_f$. For a real risky bond using the same formulae (2,3,9) and the same discount rate $r_f$ one obtains distributions of bond value $V$ represented in figure 2.

The graph is drawn for 10¼%, 10 - year bond of Cabot Corporation at time of its issuance (December 15, 1987). Bond rating is Baa1; risk–free interest rate on 10-year bonds in December 1987 was 8,99%. In this case *NPV* of the bond, if it does not default, is 108,1% of face value. Mean *NPV* is *101,48%* of face value that is near to *NPV* of risk free bond (100%).

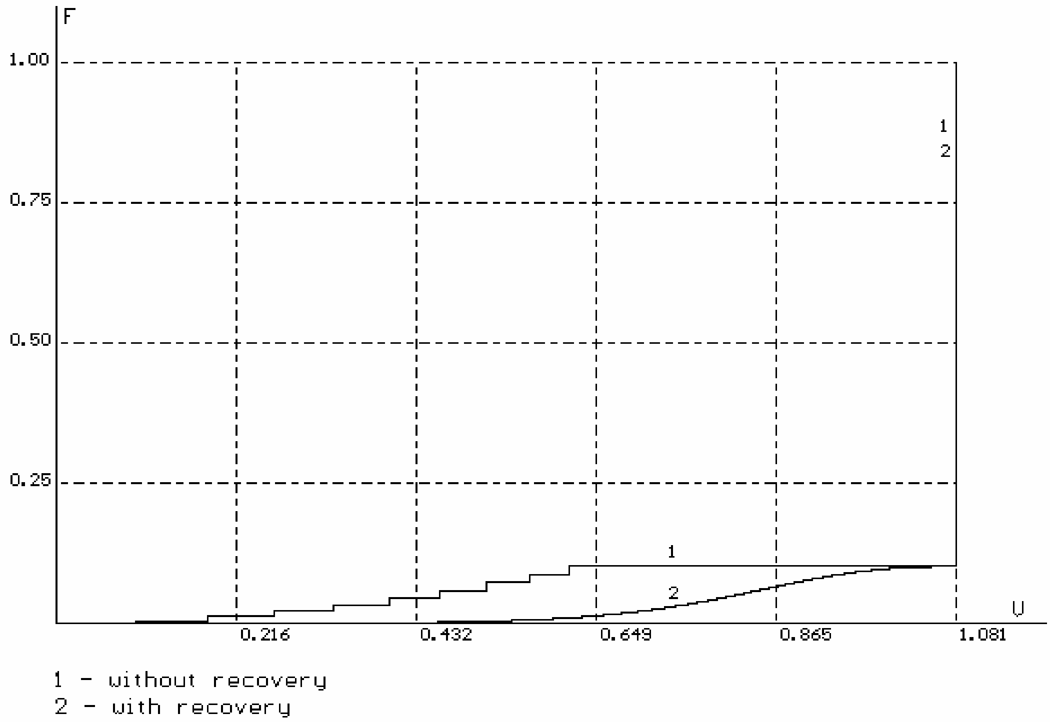


Figure 2. Cumulative distributions of NPV of 10¼% ten year bonds of Cabot Corporation (Rating Baa1) at time of issuance (December 15,1987). Recovery parameters of the bond are hypothetic.

Note that above inference contrasts with Merton's theory of valuation of risky bonds. This theory considers a bond value as the deterministic *(non-random)* function of value of the firm and time, while in the current study bond value is random variable.

### 3. Assessing probabilities $P_D(T_k)$.

While assessing probabilities $P_D(T_k)$ one must distinguish between ex-ante (unconditional) and ex-post (conditional) probabilities. An intermediate case can be also identified.

The most suitable ex-ante probabilistic characteristic of default is the default rate $\lambda$, which is defined as the conditional probability of a firm's default prior to the end of the time interval of interest, given that it was a going concern at the beginning of the time interval. In practice this probability may be determined approximately as the percentage of firms operating at the beginning of the time interval, which defaulted during that time interval

Given default rates $\lambda_1,...\lambda_n$ for a set of time intervals between successive interest payments one can calculate probabilities of corporate default during each of these intervals.

The probability of a firm's default exactly during the $k$ *-th* interval may be calculated as

$$P_D(T_k) = (1-\lambda_1)\cdot ... \cdot(1-\lambda_{k-1})\cdot\lambda_k\,, \qquad (10)$$

and the probability of default within $M+1$-th interval is

$$P_D(T_{M+}) = (1-\lambda_1)\cdot ... \cdot(1-\lambda_M)\,. \qquad (11)$$

Note that in popular semiannual interest payment schedules number of days between successive interest payments varies; equally vary default rates $\lambda_i$ for those intervals. They can be calculated via compounding of daily default rates $\lambda_d$, by means of formula:

$$\lambda_i = 1-(1-\lambda_d)^{n_i} \approx \lambda_d \cdot n_i\,,$$

where $n_i$ is number of days within $I$ - th time interval.

If default rates are determined on economy-wide (country-wide) level and averaged over the long time periods one can refer to probabilities $P_D(T_k)$ as *unconditional*.

Being based on default statistics collected from restricted subsets of firms, default rate $\lambda$ can reflect current characteristics of the macroeconomic environment depending on its fundamental (primary) parameters such as economic growth, money supply, capital market activity, new business formation rate (Altman, 1982), line of corporate business is also of principal importance.

Increased default rates for specific subsets of firms reflect interdependence (correlation) between their defaults.

Conditional probabilities $P_D(T_k\,/\,\boldsymbol{f})$ can be obtained with accounting for current financial position of a borrower (a firm). They are much more preferable than unconditional ones because enable assessing individual risk of each borrower.

Detailed description of methodology of calculation of probabilities $P_D(T_k\,/\,\boldsymbol{f})$ conditional on the set (vector) $\boldsymbol{f}$ of current financial indices of a borrower is represented in L. Philosophov, V. Philosophov (2002) and L. Philosophov, J. Batten, V. Philosophov (2003). The methodology is based on enhanced Bayesian multi-alternative models that account for non-normality and interdependence of predictive variables.

Numerical data in the studies relate to a firm's bankruptcy (identified as event of filing bankruptcy petition). Default is slightly different concept but the methodology is fully applicable. Quantitative data seem to be applicable also, though some further specification is desirable. The difficulty consists in fact that data on firms' defaults is less publicly available.

The proper choice of financial indices $\boldsymbol{f}$ is of great importance because informative prognostic variables enhance default prediction and increase accuracy of assessing default risk characteristics. L. Philosophov, J. Batten, V. Philosophov (2003) proposes two groups of predictive variables.

Group of indices of a firm's current financial position includes four financial ratios:

- $f_1$ - Working Capital / Total Assets;
- $f_2$ - Retained Earnings / Total Assets;
- $f_3$ - Earnings Before Interest and Tax / Total Assets;
- $f_4$ - Interest Payments / Total Assets.

Good predictive power of above ratios was confirmed by parts in many studies starting from Altman (see Altman 1982).

Another group of indices is first proposed in L. Philosophov, J. Batten, V. Philosophov (2003). It is derived from a schedule of paying off a firm's long-term debt and includes:

- $g_1$ - portion of long-term debt due within the first year starting from the date of the firm's last financial statements;
- $g_2$ - portion of long-term debt due within the second year;
- ………………………………………………………………
- $g_m$ - portion of long-term debt due within the $m$-th year.

The study found that in many cases repaying of large portions of long-term debt causes or triggers bankruptcy of a debtor. As a schedule of repaying long-term debt is known long ahead (for 10, 30, 40 and even 100 years), the above factors can enhance bankruptcy prediction at distant time horizons.

Relevant literature studies also variables of market anticipation of bankruptcy of a publicly traded firm (among them familiar Distance to Default). They are not studied in above cited papers but can be easily incorporated in the proposed models.

***4. Multi - period Bayesian valuation of risky debt: empirical testing.***

In this section we define "fair" interest rates on risky bonds and "fair" yield spreads on those bonds. Then those values are calculated for some real bonds and compared with observed interest rates, prices and credit spreads. In contrast with many other studies we distinguish between fair interest rates and spreads. For BMP model is important that the former are established by emitters directly at time of bond issuance while the later are established by market via bond market prices. Procedures of determining fair interest rates and yield spreads are different.

*4.1. A Risky Bond at Time of Issuance.*

Modern financial theory and practice establish risk premium $\delta r_b = r_b - r_f$. in dependence on bond rating and basing on empirical analysis of current (at time of issuance) yields on traded bonds of the same maturity and risk.

We can now propose a method of theoretical assessment of fair risky interest rate $r_b^*$. In general one can consider as fair, rate that equalizes mean NPV of a risky bond and NPV of risk-free bond with the same terms of issuance - date, maturity, dates of interest payments, etc. At the time of issuance both must be equal to a bond's par value (this is acknowledged point of view (Hull (2003)).

Following this criterion one can determine fair rate $r_b^*$ by deciding equation

$$U = V_{M+} \times P_D(T_{M+}) + \sum_{m=1}^{M} V_m \times P_D(T_m), \qquad (12)$$

where $V_m$ and $V_{M+}$ are determined in accordance with (2) and depend on $r_b^*$.

As the example in the table 1 we represent true and fair (calculated) interest rates on various senior unsecured bonds of USA firms.

| Firm | Bond rating | Date of issuance | Maturity date | Risk-free interest rate (%) | Stated interest on bond(%) | True spread | Fair interest rate (%) | Fair interest rate spread to true (%) |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| Johnson & Johnson | Aaa | 15.05.2003 | 15.05.2033 | 4,700 | 4,950 | 0,250 | 5,225 | 210,00 |
| General Electric Company | Aaa | 01.02.2003 | 01.02.1013 | 4,010 | 5,000 | 0,990 | 4,606 | 60,202 |
| Procter & Gamble | Aa3 | 15.12.2003 | 15.12.2015 | 4,460 | 4,850 | 0,390 | 5,036 | 147,94 |
| Procter & Gamble | Aa3 | 01.02.2004 | 01.02.2034 | 5,020 | 5,500 | 0,480 | 5,626 | 126,25 |
| | | | | | | | | |
| Abbott Laboratories | A1 | 15.03.2004 | 15.03.2011 | 3,230 | 3,750 | 0,520 | 3,716 | 93,462 |
| Abbott Laboratories | A1 | 15.03.2004 | 15.03.2014 | 3,750 | 4,350 | 0,600 | 4,286 | 89,333 |
| Anheuser Busch Companies . | A1 | 15.04.2004 | 15.04.2012 | 3,890 | 4,700 | 0,810 | 4,595 | 87,037 |
| Anheuser Busch Companies | A1 | 15.10.2003 | 15.10.2016 | 4,545 | 5,050 | 0,505 | 5,051 | 100,198 |
| Anheuser Busch Companies | A1 | 15.11.2005 | 15.11.2017 | 4,658 | 5,491 | 0,833 | 5,156 | 59,784 |
| Anheuser Busch Companies | A1 | 01.03.2004 | 01.03.2019 | 4,465 | 5,000 | 0,535 | 4,953 | 91,215 |
| Bristol Myers Squibb Co. | A1 | 01.10.2001 | 01.10.2011 | 4,660 | 5,750 | 1,090 | 5,146 | 44,587 |
| Bristol Myers Squibb Co. | A1 | 15.11.1996 | 15.11.2026 | 6,450 | 6,800 | 0,350 | 7,001 | 157,43 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Archer Daniels Midland Co. | A2 | 15.12.1997 | 15.12.2027 | 6,070 | 6,750 | 0,680 | 6,700 | 92,647 |
| Mean A | | | | | | | | 90,632 |
| AOL Time Warner Inc. | Baa1 | 15.04.2001 | 15.04.2011 | 5,080 | 6,750 | 1,670 | 5,931 | 50,958 |
| AOL Time Warner Inc. | Baa1 | 15.04.2001 | 15.04.2031 | 5,590 | 7,625 | 2,035 | 6,346 | 37,150 |
| AOL Time Warner Inc. | Baa1 | 01.05.2002 | 01.05.2012 | 5,130 | 6,875 | 1,745 | 6,049 | 52,665 |
| AOL Time Warner Inc. | Baa1 | 01.05.2002 | 01.05.2032 | 5,790 | 7,700 | 1,910 | 6,585 | 41,623 |
| International Paper Co. | Baa2 | 15.01.2004 | 15.01.2009 | 3,240 | 4,250 | 1,010 | 3,965 | 71,782 |
| International Paper Co. | Baa2 | 01.04.2004 | 01.04.2010 | 2,975 | 4,000 | 1,025 | 3,722 | 72,878 |
| International Paper Co. | Baa2 | 15.01.2004 | 15.01.2014 | 4,270 | 5,500 | 1,230 | 4,997 | 59,106 |
| International Paper Co. | Baa2 | 01.04.2004 | 01.04.2016 | 3,940 | 5,250 | 1,310 | 4,666 | 55,488 |
| Motorola Incorporated | Baa2 | 15.11.2000 | 15.11.2010 | 5,850 | 7,625 | 1,775 | 6,489 | 36,000 |
| Goodrich Corporation | Baa3 | 15.04.1998 | 15.04.2008 | 5,550 | 6,450 | 0,900 | 6,126 | 64,000 |
| Goodrich Corporation | Baa3 | 15.04.1998 | 15.04.2038 | 5,930 | 7,000 | 1,070 | 6,468 | 53,333 |
| Goodrich Corporation | Baa3 | 15.05.1999 | 15.05.2009 | 5,530 | 6,600 | 1,070 | 6,232 | 65,607 |
| Mean Baa | | | | | | | | 55,049 |

Table 1. True and fair interest rates on corporate bonds.

Risk-free interest rates in the table are weekly interest rates related to the last Friday before bond issuance date; data is taken from USA Federal Reserve statistical releases.

Mean annual ex-ante default rate $\lambda = 1,26\%$ used in calculation of probabilities $P_D(T_m / \boldsymbol{f})$ and mean annual recovery rate $\beta = 44,9\%$ used in calculation of $V_m$ are taken from extensive study of Moody's Investor Service, (2005). They cover all rated bonds observed in time interval 1970 – 2004.

According to the table 1 default risk (as assessed by BMP model) explains on the average about 90% of observed interest rate spreads for *A* rated bonds and about 55% for bonds rated *Baa.* We assess such coincidence of true and fair interest rates as good.

Waiting for still better coincidence is problematic because of some indefiniteness that is present in setting risk-free interest rates, mean ex-ante default and recovery rates. This influences calculated fair interest rates (most sensitive to improperly set risk-free interest rates are bond with high ratings Aaa, Aa).

Another source of discordance consists in fact that one can use various sets of probabilities $P_D(T_m)$ for calculating fair risky interest rates. Those can be ex-ante probabilities calculated via annual default rates and several variants of ex-post probabilities $P_D(T_m / \boldsymbol{f})$, conditional on various sets $\boldsymbol{f}$ of indices of a firm's current financial position, as described in section 3. Different sets of variables $\boldsymbol{f}$ informative in predicting defaults, will lead to different estimators $r_b^*$. Coincidence with observed interest rates will be better if set $\boldsymbol{f}$ is near to that implicitly used by market, rating agencies, investment banks.

Note also that even after adjusting mean NPV of risky bond original NPV stays to be random, and investors can demand additional compensation for this randomness.

Data of the Table 1 contrasts with numerous studies (Elton, Gruber, Agrawal, Mann 2001; Huang, Huang 2003; Longstaff, Mittal, Neis 2005; Driessen 2005; Chen, Collin-Dufresne, Goldstein 2005; Crermers, Driessen, Maenhout, Weinbaum 2005; Berndt, Douglas Duffie, Ferguson, Schranz 2005) that used Merton type bond valuation models and found that default risk explains only small part of observed credit spreads (5 through 22 percents according to Delianedis and Geske (2001)). The rest those studies attribute to taxation differences between corporate and Treasury bonds, jumps in asset value process, liquidity and market effects.

Current study represents one more simple explanation of above findings: this is extreme imprecision of Merton's model.

*4.2. A Risky Bond at an arbitrary time $t$.*

Consider now the same risky bond at an arbitrary time $t$ of its life cycle. Suppose time $t$ is within interval $T_j$, i.e. between $t_{j-1}$ and $t_j$ moments of interest payment.

The bond's NPV stays to be random but number of its (discrete) possible values decreases because some horizons of interest payments are already passed.

Those values can be described using expressions similar to that of formulae (2):

- $V^{(t)} = V_j^{(t)} = 0$ with probability $P_D^{(t)}(T_j^{(t)})$;
- $V_{j+1}^{(t)} = \frac{r_b^{(j)}U}{(1+r_f^{(j)})}$ with probability $P_D^{(t)}(T_{j+1})$;
- 
- $V_m^{(t)} = \frac{r_b^{(j)}U}{(1+r_f^{(j)})} + \sum_{k=j+1}^{m-1} \frac{r_bU}{(1+r_f^{(j)})(1+r_f)^{k-j}}$ with probability $P_D^{(t)}(T_m)$; (13)
- ………………………………………………………………
- $V_{M+}^{(t)} = \frac{r_b^{(j)}U}{(1+r_f^{(j)})} + \sum_{k=j+1}^{M} \frac{r_bU}{(1+r_f^{(j)})(1+r_f)^{k-j}} + \frac{U}{(1+r_f^{(j)})(1+r_f)^{M-j}}$ with probability $P_D^{(t)}(T_{M+})$.

Placed separately in expressions (13) are addendums corresponding to the time interval $T_j^{(t)}$ between $t$ and $t_j$ that constitutes only part of standard time interval between interest payments. For this interval one need apply corrected interest and discount rates $r_b^{(j)}, r_f^{(j)}$ and corrected probability of default $P_D^{(t)}(T_j^{(t)})$. Generally $P_D^{(t)}(T_m)$ denote probabilities of a firm's default within time intervals $T_m$ as they can be assessed at time $t$.

Risk-free interest rate $r_f$ is also considered for time $t$.

Cumulative distribution of a bond value is now:

$$F(V) = P_D^{(t)}(T_{M+}) \times 1(V - V_{M+}^{(t)}) + \sum_{m=j}^{M} P_D^{(t)}(T_m) \times 1(V - V_m^{(t)}) , \quad (14)$$

and mean value of a bond can be determined using formula similar to formula (5):

$$\overline{V}^{(t)} = V_{M+}^{(t)} \times P_D^{(t)}(T_{M+}) + \sum_{m=j}^{M} V_m^{(t)} \times P_D^{(t)}(T_m) . \quad (15)$$

Formulae (6,7,8) related to non-zero recovery rates can be rewritten for time $t_j$ by analogy.

Considering risky bonds at an arbitrary time $t$ one can try to explain observed marked prices and observed yields on those bonds and determine what influences their values. It is interesting to establish if risk of bond's default is the only or at least principal factor that determines observed credit spreads.

We can do this by introducing concept of "fair prices" and "fair yields" on risky bonds.

Fair price of bond $V^*$ can be considered to be equal to bond's mean NPV - $\overline{V}^{(t)}$ as determined by (15).

Yield to maturity $Y$ is usually determined as a decision of equation, which in our notation can be written as:

$$V^{(mr)} = \frac{r_b^{(j)} U}{(1 + Y^{(j)})} + \sum_{k=j+1}^{M} \frac{r_b U}{(1 + Y^{(j)})(1 + Y)^{k-j}} + \frac{U}{(1 + Y^{(j)})(1 + Y)^{M-j}} \quad (16)$$

where $V^{(mr)}$ is market price of the bond at time moment $t$ , $r_b^{(j)}$ and $Y^{(j)}$ are interest rate $r_b$ and yield $Y$ recalculated to partial time interval.

For a risky bond equation (16) implies that a bond survives until maturity and brings increased (risky) yield relatively to risk-free yield (interest rate $r_f$ ) as compensation for possibility of default.

Really, possibility of default within one of future time intervals $T_m$ leads to random bond value $V_m^{(t)}$, as described by formulae (13), and randomness of true yield (yield to default). Possible values of random yield can be found by deciding the set of equations $V^{(mr)} = V_m^{(t)}(Y)$, for $m = j,...M,M+$ , where $V_m^{(t)}(Y)$ are taken from (13) with $Y$ used as substitute for $r_f$ . Minimally possible yield is equal –100% and corresponds to situation when investment is fully lost, i.e. a firm defaults before any interest is paid and there is no recovery.

To clarify, in table 2 we represent data of Abbott Laboratories (ABT) whose senior unsecured bond issued on the March 15, 2004 is traded on the December 22, 2006. The bond pays semiannual interest at annual rate 4,35% and matures on the March 15, 2014.

Moody's rating of the bond is A1, market price as of December 22, 2006 is 96,680% of par and stated yield to maturity 4,901%.

Rows of the table correspond to semiannual intervals between successive interest payments. Columns 2, 3 represent beginning and ending dates of each interval; observation date (December 22, 2006) is within the sixth interval. The last - 21th interval covers all dates after the end of bonds' life period.

Probabilities of default for Abbott Laboratories for each time interval are represented in column 4. They were calculated as described in section 3 basing on Abbott Laboratories accounting data of as of December 31, 2005. As before, annual ex-ante default rate was taken equal to 1,26% and recovery rate to 44,9%.

| Interval # | Interval start date | Interval end date | Probability. of default | Yield to default (%) |
|---|---|---|---|---|
| 6 | 22.12.2006 | 15.03.2007 | 0,0003 | -7,422 |
| 7 | 15.03.2007 | 15.09.2007 | 0,0008 | -7,339 |
| 8 | 15.09.2007 | 15.03.2008 | 0,0018 | -7,146 |
| 9 | 15.03.2008 | 15.09.2008 | 0,0033 | -6,944 |
| 10 | 15.09.2008 | 15.03.2009 | 0,0040 | -6,733 |
| 11 | 15.03.2009 | 15.09.2009 | 0,0051 | -6,512 |
| 12 | 15.09.2009 | 15.03.2010 | 0,0054 | -6,283 |
| 13 | 15.03.2010 | 15.09.2010 | 0,0061 | -6,046 |
| 14 | 15.09.2010 | 15.03.2011 | 0,0064 | -5,801 |
| 15 | 15.03.2011 | 15.09.2011 | 0,0071 | -5,549 |
| 16 | 15.09.2011 | 15.03.2012 | 0,0073 | -5,291 |
| 17 | 15.03.2012 | 15.09.2012 | 0,0077 | -5,028 |
| 18 | 15.09.2012 | 15.03.2013 | 0,0068 | -4,761 |
| 19 | 15.03.2013 | 15.09.2013 | 0,0060 | -4,491 |
| 20 | 15.09.2013 | 15.03.2014 | 0,0058 | -4,219 |
| 21 | 15.03.2014 | all later dates | 0,9261 | 4,901 |

Table 2. Yields on bond in dependence on time interval of its default. The data corresponds to 4,35% 10-year senior unsecured bond of Abbott Laboratories due March 15, 2014. Bond is traded on the December 22, 2006 at price 96,680%. Its stated yield to maturity is 4,901%.

The column 5 represents yields to default calculated by means of formulae (13) for situations of possible ABT default within each time interval. All defaults within active bond period lead to loss of principal and as a consequence to negative yield. If default does not occur or occurs after bond's active period, yield to default is the same as yield to maturity and is equal to 4,901%. This amount is exactly the same as was achieved in actual trades. Table 3 evidences that such yield can be obtained with probability 0,9261.

In general data of the table 2 describes probabilistic distribution of random yield; the distribution is of discrete type.

To explain observed market yields to maturity on risky bonds theoretically (i.e. to calculate "fair yields") one can act in two different ways.

1. Determine fair price of risky bond by means of formula (15) and then use it in equation (16) instead of market price to determine "fair" risky yield.

2. Try to directly infer 'fair" risky yields from risk-free yield and probabilities of default. There are several variants of such inference; the most simple and transparent is to suppose that average of risky market yields (like those represented in table 2) must be near to risk-free yield.

$$Y_{M+}^{(t)} \times P_D^{(t)}(T_{M+}) + \sum_{m=j}^{M} Y_m^{(t)} \times P_D^{(t)}(T_m) \approx Y_{rf}^{(t)} .$$

Considering further that all negative yields are inappropriate and have restricted economic sense we can equalize them to zero. As a result we obtain for yield to maturity $Y_{M+}^{(t)}$:

$$Y_{M+}^{(t)} \times P_D^{(t)}(T_{M+}) = Y_{rf}^{(t)} \quad \text{and} \quad Y_{M+}^{(t)} = Y_{rf}^{(t)} / P_D^{(t)}(T_{M+}) . \qquad (15)$$

The resulting formula is simple and rather transparent. Risky yield to maturity must exceed risk-free yield the more, the less is probability $P_D^{(t)}(T_{M+})$ (i.e. the more is probability $1 - P_D^{(t)}(T_{M+})$ that default does occur within active bond period).

Table 3 represents examples of calculation of fair prices and fair yields to maturity for some issues of senior unsecured bonds of USA industrial firms. Data on bond issues and issuers' accounting information is taken from SEC filings and annual corporate reports. Data on bond market prices and yields to maturity is NASD data on OTC trades.

Represented in the table (last four columns) are Fair Prices, Fair Yields, calculated by two just described methods and ratio of fair to observed yield spreads (for fair yields calculated via fair price). Risk-free interest rates in the table are weekly interest rates related to the last Friday before bond trade date; data is taken from USA Federal Reserve statistical releases.

Mean annual default rate ($\lambda = 1,26\%$) and mean annual recovery rate ($\beta = 44,9\%$) used in calculation are taken as before from extensive study of Moody's Investor Service, (2005).

| Firm | Bond rating | Bond matures | Bond interest rate | Trade Date | Risk-free Rate (weekly) | Closing Price | Closing Yield | Fair price | Fair yield (as determined by fair price) | Fair yield (as determined directly) | Fair yield spread to true yield spread (%) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Abbott Laboratories | A1 | 15.03.2011 | 3,750 | 15.09.2006 | 4,735 | 93,863 | 5,305 | 94,466 | 5,145 | 4,898 | 71,928 |
| Abbott Laboratories | A1 | 15.03.2011 | 3,750 | 22.12.2006 | 4,5793 | 94,680 | 5,200 | 95,230 | 5,017 | 4,737 | 74,337 |
| Abbott Laboratories | A1 | 15.03.2014 | 4,350 | 14.09.2006 | 4,748 | 93,100 | 5,486 | 94,217 | 5,295 | 5,128 | 74,108 |
| Abbott Laboratories | A1 | 15.03.2014 | 4,350 | 22.12.2006 | 4,572 | 96,680 | 4,901 | 95,270 | 5,142 | 4,937 | 173,420 |
| Anheuser Busch Companies Inc. | A1 | 15.04.2012 | 4,700 | 22.12.2006 | 4,562 | 98,322 | 5,064 | 98,939 | 5,053 | 4,699 | 97,889 |
| Anheuser Busch Companies Inc. | A1 | 15.10.2016 | 5,050 | 15.09.2006 | 4,792 | 96,345 | 5,528 | 98,501 | 5,244 | 5,236 | 61,455 |
| Anheuser Busch Companies Inc. | A1 | 15.10.2016 | 5,050 | 22.12.2006 | 4,598 | 96,850 | 5,468 | 99,894 | 5,064 | 5,023 | 53,555 |
| Anheuser Busch Companies Inc. | A1 | 01.03.2019 | 5,000 | 15.09.2006 | 4,844 | 96,889 | 5,345 | 97,030 | 5,330 | 5,454 | 96,986 |
| Anheuser Busch Companies Inc. | A1 | 01.03.2019 | 5,000 | 21.12.2006 | 6,648 | 94,397 | 5,641 | 98,686 | 5,147 | 5,233 | 50,251 |
| Bristol Myers Squibb Co. | A1 | 01.10.2011 | 5,750 | 15.09.2006 | 4,730 | 102,705 | 5,133 | 103,395 | 4,981 | 4,844 | 62,263 |
| Bristol Myers Squibb Co. | A1 | 01.10.2011 | 5,750 | 21.12.2006 | 4,566 | 101,250 | 5,446 | 103,884 | 4,830 | 4,674 | 30,035 |
| Bristol Myers Squibb Co. | A1 | 15.11.2026 | 6,800 | 13.09.2006 | 5,009 | 107,790 | 6,121 | 115,312 | 5,531 | 6,216 | 46,948 |
| Bristol Myers Squibb Co. | A1 | 15.11.2026 | 6,800 | 22.12.2006 | 4,818 | 110,744 | 5,876 | 117,591 | 5,353 | 5,977 | 50,583 |
| Mean for A1 bonds | | | | | | | | | | | 67,662 |
| International Paper Co. | Baa2 | 15.01.2009 | 4,250 | 15.09.2006 | 4,790 | 97,725 | 5,302 | 97,381 | 5,460 | 4,926 | 130,775 |
| International Paper Co. | Baa2 | 15.01.2009 | 4,250 | 22.12.2006 | 4,694 | 96,547 | 6,068 | 97,734 | 5,425 | 4,822 | 53,219 |
| International Paper Co. | Baa2 | 01.04.2010 | 4,000 | 15.09.2006 | 4,745 | 94,125 | 5,866 | 95,224 | 5,502 | 4,980 | 67,541 |
| International Paper Co. | Baa2 | 01.04.2010 | 4,000 | 22.12.2006 | 4,603 | 95,845 | 5,406 | 95,830 | 5,407 | 4,826 | 100,175 |
| International Paper Co. | Baa2 | 01.09.2011 | 6,750 | 21.12.2006 | 4,568 | 106,759 | 5,110 | 105,789 | 5,340 | 4,876 | 142,432 |
| International Paper Co. | Baa2 | 15.01.2014 | 5,500 | 15.09.2006 | 4,746 | 98,088 | 5,823 | 100,046 | 5,493 | 5,243 | 69,371 |
| International Paper Co. | Baa2 | 15.01.2014 | 5,500 | 21.12.2006 | 4,571 | 99,712 | 5,549 | 100,928 | 5,341 | 5,045 | 78,731 |
| International Paper Co. | Baa2 | 01.04.2016 | 5,250 | 14.09.2006 | 4,782 | 93,970 | 6,093 | 98,041 | 5,517 | 5,434 | 56,049 |
| International Paper Co. | Baa2 | 01.04.2016 | 5,250 | 20.12.2006 | 4,593 | 96,188 | 5,786 | 99,311 | 5,346 | 5,213 | 63,124 |
| Goodrich Corporation | Baa3 | 15.04.2008 | 6,450 | 14.09.2006 | 4,897 | 99,600 | 6,718 | 101,648 | 5,353 | 4,961 | 25,037 |
| Goodrich Corporation | Baa3 | 15.04.2038 | 7,000 | 28.08.2006 | 4,950 | 103,650 | 6,719 | 120,322 | 5,619 | 7,399 | 37,818 |
| Goodrich Corporation | Baa3 | 15.05.2009 | 6,600 | 15.09.2006 | 4,770 | 103,109 | 5,324 | 103,014 | 5,373 | 4,911 | 108,847 |
| Mean for Baa bonds | | | | | | | | | | | 77,760 |

Table 3. True and fair prices and yields to maturity of some issues of US corporate bonds.

Table 3 evidences that, like interest rate spreads in table 1, default yield spreads constitute significant part of total credit yield spreads, on average nearly 72%. This again contrasts with what can be obtained from structural models. At the same time other factors, discussed in section 4.1 (premium for randomness of bond's NPV, taxation differences between corporate and Treasury bonds, liquidity and market effects), also contribute to observed credit spreads. Note also that fair yields, calculated via fair price are nearer to observed risky yields to maturity than directly calculated fair yields.

### *4. Conclusion*.

Described model for valuation risky bond and loans can be most conveniently realized as a PC program. The model has many facilities for natural mapping it to each specific bond, loan, firm, macroeconomic situation. We do not see any drawbacks or obstacles that can impede its practical application, though much wider testing must precede this application.

Note once more that the model does not use any simplifying assumptions. It does not suppose normality of predictive variables or their mutual independence. We do not use any predetermined model of ex-post default probabilities (conditional hazard rates) or of future dynamics of predictive variables. We calculate ex-post probabilities within enhanced Bayesian methodology; empirical data is used in intermediate stages of the methodology to assess conditional distributions of predictive variables.

Literature.


Altman E., 1982. Corporate Financial Distress, a Complete Guide to Predicting, Avoiding and Dealing with Bankruptcy, (John Wiley, New York.).

Berndt A., Douglas R., Duffie D., Ferguson M., Schranz D., 2005, Measuring Default Risk Premia from Default Swap Rates and EDF's, Working Paper.

Bharath S., Shumway T., 2004, Forecasting Default with the KMV-Merton Model, Working paper, University of Michigan.

Black F., Scholes M. 1973, The Pricing of Options and Corporate Liabilities, Journal of Political Economy, 81, 637-654.

Bohn J., 2000, A Surveyof Contingent-Claims Approaches to Risky Debt Valuation, Institutional Investor Inc.

Brigham E.F., 1991, Fundamentals of Financial Management, Dryden Press.

Chen L., Collin-Dufresne P., Goldstein R., 2005, On the relation Between CreditSpread Puzzles and Equity Premium Puzzle, Working Paper, Michigan State University , U.C. Berkeley, and University of Minnesota.

Collin-Dufresne P., Goldstein R., Martin S., 2001, The determinants of Credit Spread Changes, Journal of Finance, 56, 2177-2208.

Credit Suisse First Boston 1997, $CreditRisk+^{TM}$ 1997, A Credit Risk Management Framework,

Cremers M., Driessen J., Maenhourt P., Weinbaum D., 2005, Individual Stock Option Prices and Credit Spreads, Working Paper, Yale University, University of Amsterdam, INSEAD, Cornell University.

Crosbie P., Bohn J.,2003, Modelling Default Risk, Moody's KMV document.

Delianedis G., Geske R., 2001, The Components of Corporate Credit Spreads: Default, Recocery, Tax, Jumps, Liquidity, and Market Factors,

Driessen J., (2005), Is Default Event Risk Priced in Corporate Bonds?, Journal of Finance, 18, 165 - 195.

Duffie D., Saita L., Wang K., 2005, Multi-period Corporate Default Prediction with Stochastic Covariates, Working paper, Stanford University.

Elton E., Gruber M., Agrawal D., Mann C., 2001, Explaining the Rate Spread on Corporate Bonds, Journal of Finance,56, 247-278.

Eom Y., J. Helwege, J-Z Huang, 2004, Structural Models of Corporate Bond Pricing: An Empirical Analysis, The Review of Financial Studies, vol. 17, pp499-544.

Feller W., 1966, An Introduction to Probability Theory and its Applications, (John Wiley & Sons, Inc., New York, London, Sydney).

Hillegeist S.A., Cram D.P., Keating E.K. Lundstedt K.G., 2002, Assessing the Probability of Bankruptcy, Working paper, Northwestern University.

Huang J.Z., Huang M.,2003, How Much of Corporate- Treasury Yield Spread is Due to Credit Risk?, Working paper, Penn State University and Stanford University.

Jarrow R., Turnbull S. 1995, Pricing Derivatives on Financial Securities Subject to Credit Risk, Journal of Finance, 50, 53-86.

Jones E., Mason S., Rosenfeld E., 1984, Contingent Claims Analysis of Corporate Capital Structures: An Empirical Investigation, Journal of Finance, 39, 611-625.

JP Morgan 1997, *CreditMerics*™ , Technical document,.

Kim I., Ramaswamy K., Sundaresan S., 1993, Does Default Risk in Coupons Affect the Valuation of Corporate Bond?, A Contingent Claims Model, Financial Management, 22, 117-131.

Leland H., 2002, Prediction of Expected Default Frequencies in Structural Models o Debt, working paper.

Longstaff F., Mittal S., Neis E., 2005, Corporate Yield Spreads: Default Risk or Liquidity? New Evidence from the Credit Default Swap Market, Journal of Finance, 60, 2213-2253.

Lyden S., Saraniti D., 2000, An Empirical Investigation of the Classical Theory of Corporate Security Valuation, Barclays Global investors.

Merton R. 1974, On the Pricing of Corporate Debt: The Risk Structure of Interest Rates, Journal of Finance, 29, 449-470.

Moody's Investor Service, 2005, Default & Recovery Rates of Corporate Bond Issuers, 1920-2004.

Ogden J., 1987, Determinants of Credit Ratings and Yields on Corporate Bonds: Test of Contingent Claims Model, Journal of Financial Research, 10, 329-339.

Philosophov L., Philosophov V., 2002, Corporate bankruptcy prognosis: An attempt at a combined prediction of the bankruptcy event and time interval of its occurrence, International Review of Financial Analysis, 11, 375-406.

Philosophov L., Batten J., Philosophov V., 2003, Assessing the Time Horizon of Bankruptcy Using Financial Ratios and the Maturity Schedule of Long-Term Debt, http://ssrn.com/abstract=461460 - November 2003, EFA-2005 (Moscow).