ASSESSING CREDIT LOSS DISTRIBUTIONS
FOR INDIVIDUAL BORROWERS AND CREDIT PORTFOLIOS.
BAYESIAN MULTI-PERIOD MODEL VS. BASEL II MODEL.

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August 9, 2004

Abstract.

As development of the New Basel Capital Accord (Basel II) approaches to its final, the problem of validation and calibration of the Basel II models attracts increasing interest of scientists and practitioners. This is evidenced by the Basel Committee press releases, recent past and forthcoming conference programs, scientific publication on the problem.

The principal attention of the Basel II is concentrated on credit risk where validation problems seem to be solvable, because credit risks of individual borrowers and portfolios are accessible for precise calculations.

The current paper assesses validity of credit loss distributions, calculated by means of the Basel II model. The assessment method consists in parallel calculations the same distributions by means of exact probabilistic formulae. The exact calculation scheme is realized within Bayesian multi-period (BMP) model and Portfolio BMP model.

We found that Basel II model ensures correct assessments of credit loss distributions for medium and large portfolios of thirty or more one-period credits (credits without intermediate interest etc. payments). For small portfolios and individual one-period credits assessments of Basel II model are rough or extremely rough. Basel II model undervalues credit risk of portfolios of multi-period credits (credits with active period covering several years and intermediate payments).

The study provides an alternative look at some principal Basel II concepts, like Probability of Default (PD) and Correlation. This can serve to more correct understanding and flexible use of those concepts in assessing credit risks.

Exact models are more complex that Base II model, but perfectly available for quick computer calculations. They can be used in bank practice ensuring flexible assessments of credit risk.

Keywords: Credit risk, Credit loss distribution, Basel II, Bayesian model.

GEL codes: C11, C13, C51, E58, G21, G33.

1. Introduction.

The Basel Committee on Banking Supervision published document entitled "International Convergence of Capital Measurement and Capital Standards" known also as the New Basel Capital Accord (Basel II). The document establishes the new framework for assessment bank capital adequacy with strong emphasis on improvement of bank capabilities to assess and manage risks. The new Accord will be put in action in OECD countries by the end of 2006 and its most complex parts y the end of 2007.

A key element of the new framework consists in calculation of bank's *risk-weighted assets*, which depend on probabilistic assessment of marginal losses that bank can bear due to influence of market, credit, operational etc. risks.

Principal attention of the New Basel Accord is focused on credit risk, which reflects bank's losses caused by clients' defaults. Along with Standardized approach to credit losses based on external credit ratings of borrowers the New Accord proposes Internal Rating-Based (IRB) approach in its foundation and advanced versions. Both versions suppose preliminary determination of four key inputs to the model that estimates required capital and risk-weighted assets. Those inputs are: probability of clients default (PD); loss given default (LGD); exposure at default (EAD); effective maturity (M). Only one of the parameters (PD) directly characterizes a specific borrower, LGD also might characterize client but in the current model this is the preset deterministic (non-random) parameter depending on the seniority of debt. EAD and M are pure characteristics of a current credit.

These inputs must be substituted in formulae represented in the New Accord to calculate required capital and risk-weighted assets. The same formulae are proposed for corporate sovereign and bank exposures as well as for client portfolios.

The question arises if the single parameter PD is sufficient characteristic of borrowers and their risks.

The current paper provides an attempt of validation of Basel II credit risk model, in application to individual borrowers and homogenous portfolios of borrowers.

To do this we develop an alternative - Bayesian Multi-period (BMP) model (based on exact probabilistic formulae) for assessing credit risk of individual borrowers and then aggregating them in homogeneous portfolios within Portfolio Bayesian Multi-period (PBMP) model. Both models use detailed description of time schedules of true cash flows between bank and borrower and calculate conditional probabilities of a borrower's default at all stages of cash flow process. The approaches are based on studies L. Philosophov, V. Philosophov (2002) and L. Philosophov, J. Batten, V. Philosophov (2003), which assess time horizons of a firm's bankruptcy basing on its current financial indices and schedule of repaying of long-term debt.

We found that Basel II model ensures correct assessments of credit loss distributions for medium and large portfolios of thirty or more one-period credits (credits without intermediate interest etc. payments). For small portfolios and individual one-period credits assessments of Basel II model are rough or extremely rough. Basel II model undervalues credit risk of portfolios of multi-period credits - credits with active period covering several years and intermediate payments.

The study provides an alternative look at some principal Basel II concepts, like Probability of Default (PD) and Correlation. This can serve to more correct understanding and flexible use of those concepts in assessing credit risks.

In line with discussing necessary changes in Basel II, the current paper proposes BMP and PBMP models as an alternative to Basel II model.

Models for assessment credit loss distributions known from literature were developed mainly by major investment banks and rating agencies. Such models of J.P. Morgan (1997) Credit Suisse First Boston (1997), Moody's KMV Corporation are categorized as commercial products and described mainly in hardly available working papers. As one can understand the models are based on various abstract indexes of a client's risk. Gordy (1998), who compared models of J.P. Morgan and Credit Suisse First Boston has found that they have similar underlying mathematical structure. According to his description both models utilize rather simplified model of bank's client and are interested in characteristics of portfolio of such clients.

Bayesian Multi-Period Model developed in this paper does not fall in any specific category of models described in Basel Committee document (1999). This is structural multi-period model; like Merton-type models it assesses probability of a firm's default at various time horizons in future, but it does not use assumption of Brownian motion of market prices of a firm's equity. In contrast it relies on fundamental characteristics of a borrower and involves in analysis its detailed financial data. This allows for calculation of ex-post default probabilities conditional on indices of firm's current financial position. This is why the model was identified as Bayesian.

The rest of the document is organized as follows.

Sections 2,3 describe the BMP model and its important component parts.

Section 4 compares principal features of BMP and Basel II model and credit loss distributions of the same single borrower calculated by those models.

Section 5 calculates exact credit loss distributions for portfolio of one-period credits (PBMP model) and compares them with the same distributions provided by Basel II model.

One-period credits are understood as credits without intermediate interest payments and other intermediate cash flows.

Section 6 calculates exact credit loss distributions for portfolio of multi-period credits (by means of PBMP model) and again compares them with Basel II distributions.

Section 7 concludes.

2. Assessing credit loss distribution for an individual borrower. A Bayesian multi-period model.

For each separate borrower we consider standard cash flow schedule where at time moment t_0 bank lends to a client U dollars. The loan must be returned after M years at time moment t_M and additionally client pays annual interest at rate r (at time moments t_m). If a client returns credit after M years, balance of discounted cash flows between bank and client is equal to that represented by formula (1):

$$B = -U + \sum_{m=1}^{M} \frac{rU}{(1+d)^m} + \frac{U}{(1+d)^M} \tag{1}$$

We choose discount rate d be equal to interest rate on debt r so that after repaying the principal debt value U, balance of cash flows between bank and a borrower is equal to zero ($B = 0$).

A client is subject to default, which occurs at unknown time moment t_D. This event can take place within:

- time interval $T_1 \Rightarrow \{t_0 \div t_1\}$ with probability $P_D(T_1)$;
- time interval $T_2 \Rightarrow \{t_1 \div t_2\}$ with probability $P_D(T_2)$;

……………………………………………………………………..

- time interval $T_M \Rightarrow \{t_{M-1} \div t_M\}$ with probability $P_D(T_M)$;
- time interval $T_{M+} \Rightarrow \{t_M \div t_\infty\}$ with probability $P_D(T_{M+})$.

All intervals except the last one are supposed to be equal to one year, though other intervals can be also considered if necessary. The last time interval corresponds to defaults, which occur after the debt is paid off. It includes situation when default does not occur; this situation is referred to as default at infinite time (t_∞). Note that $P_D(T_{M+}) = 1 - P_D(T_1) - .... - P_D(T_M)$.

If a client defaults before the debt is paid off, cash flows between bank and borrower cease and balance becomes negative. The bank bears losses $L = -B$.

One can see from (1) that loss can take one of the discrete values, namely:

- loss L is equal to U ($L_1 = U$) with probability $P_D(T_1)$;
- ……………………………………………………………………
- loss is equal to $L_m = U - \sum_{j=1}^{m=1} \frac{rU}{(1+i)^j}$ with probability $P_D(T_m)$;
- ……………………………………………………………………
- loss is equal to $L_{M+} = 0$ with probability $P_D(T_{M+})$. (2)

Cumulative loss distribution function $F(L)$ is a stepwise function. Steps occur at values $L = L_m$ and have magnitudes $P_D(T_m)$:

$$F(L) = P_D(T_{M+}) \cdot 1(L) + \sum_{m=1}^{M} P_D(T_m) \cdot 1(L - L_m), \tag{3}$$

where function $1(x)$ is defined as: $1(x) = 0$ if $x < 0$ and $1(x) = 1$ if $x \geq 0$.

Note that what we just obtained is not *Loss Given Default* but unconditional loss distribution function where default probabilities are already incorporated in loss distribution. Unconditional losses are of final interest to bank.

One can see that unconditional loss distribution function admits possibilities of zero and non-zero losses.

The first addendum in the right-hand side expression in (3) corresponds to finite probability of zero losses. The loss is zero if default does not occur or occurs after active credit period.

Possibility of zero losses is absent in the current Basel II model.

The second addendum in (3) characterizes distribution of non-zero losses.

Loss given default *(LGD)* is also random within this model because defaults at different stages of credit process lead to different losses. Its cumulative distribution is (*LGD* is measured as a percent of initial exposure):

$$F(LGD) = (1 - P_D(T_{M+}))^{-1} \cdot \sum_{m=1}^{M} P_D(T_m) \cdot 1(LGD - L_m / U). \tag{4}$$

One can also calculate that mean loss is

$$\overline{L} = \sum_{m=1}^{M} P_D(T_m) \cdot L_m, \tag{5}$$

while mean Loss Given Default is

$$\overline{LGD} = \frac{1}{1 - P_D(T_{M+})} \cdot \sum_{m=1}^{M} P_D(T_m) \cdot L_m / U. \tag{6}$$

Suppose now that after a defaulting client's recovery procedures are over, bank receives back part β of lost amount. If β is fixed, bank's losses L_m (if they are nonzero) are reduced to $L_m(1-\beta)$. One can obtain now cumulative loss distribution function given recovery rate β as:

$$F(L/\beta) = P_D(T_{M+}) \cdot 1(L) + \sum_{m=1}^{M} P_D(T_m) \cdot 1(L - L_m(1-\beta)). \tag{7}$$

Cumulative distribution of LGD is now

$$F(LGD/\beta) = (1 - P_D(T_{M+}))^{-1} \cdot \sum_{m=1}^{M} P_D(T_m) \cdot 1(LGD - L_m(1-\beta)/U). \tag{8}$$

Though it is not explicitly stated, Basel II considers nonzero but nonrandom recovery rates. They are introduced by constant values depending on a seniority of debt. Comparing (7, 8) with (3, 4), one can see that such rates compress graphs of cumulative loss distributions $1-\beta$ times along horizontal axis; this holds for Basel II credit risk model also. Taking this in account we can further consider zero recovery rates $(\beta = 0)$ only.

3. Assessing probabilities $P_D(T_m)$.

Methodology of assessing probabilities $P_D(T_m)$ of a firm's default within any given time intervals was developed in the studies L. Philosophov, V. Philosophov (2002) and L. Philosophov, J. Batten, V. Philosophov (2003). Numerical data in the studies relate to a firm's bankruptcy (identified as event of filing a bankruptcy petition). Default is slightly different concept but the methodology is fully applicable. Quantitative data seem to be applicable also, though some further specification is desirable. The difficulty consist in fact that data on firms' defaults is less publicly available, through this difficulty does not relate to banks and their clients.

While assessing probabilities $P_D(T_m)$ one must distinguish between ex-ante (unconditional) and ex-post (conditional) probabilities. An intermediate case can be also identified.

The most suitable ex-ante probabilistic characteristic of default is the default rate π, which is defined as the conditional probability of a borrower's default prior to the end of the time interval of interest (usually one year), given that it was acting at the beginning of the time interval. In practice this probability may be determined approximately as the percentage of clients operating at the beginning of the time interval, which defaulted during that time interval

Given default rates $\pi_1,...\pi_m...$ for a successive number of time intervals (years) one can calculate the probabilities of client's default during each of these years.

The probability of a client's default during the m-*th* year can be calculated as

$$P_D(T_m) = (1-\pi_1)\cdot ... \cdot (1-\pi_{m-1})\cdot \pi_m, \tag{9}$$

and the probability of default within $M+1$-th year or later is

$$P_D(T_{M+}) = (1-\pi_1)\cdot ... \cdot (1-\pi_M). \tag{10}$$

If default rate is constant,

$$P_D(T_m) = (1-\pi)^{m-1}\cdot \pi, \tag{11}$$

$$P_D(T_{M+}) = (1-\pi)^M. \tag{12}$$

Models of this type are very popular in Probability Theory. Their underlying assumption of the independence of default events means that the distribution of time remaining until default does not (ex-ante) depend on client's history, i.e. on how long it was operating as an active borrower before, Feller (1966).

If default rates are determined on economy-wide (country-wide) level and averaged over the long time periods one can refer to probabilities $P_D(T_m)$ as *unconditional*.

Being based on default statistics collected from restricted subsets of borrowers, default rate π can reflect current characteristic of the macroeconomic environment depending on its fundamental (primary) parameters. Altman (1982) identified four parameters influencing the bankruptcy (business failure) rate - economic growth activity, money supply, capital market activity, new business formation rate. Line of corporate business is also of principal importance for default rate.

Increased default rates for specific subsets of bank clients reflect interdependence (correlation) between their defaults.

Within terminology of Basel (1999) document probabilities $P_D(T_m)$ determined by means of (9, 10, 11, 12) for restricted subsets of firms must be qualified as conditional, though we would like to refer them as semi-conditional.

Truly conditional probabilities can be obtained with accounting for current financial position of each (corporate) client. They are much more preferable than unconditional ones because enable assessing individual risk of each borrower.

Detailed description of methodology of calculation of probabilities $P_D(T_m / \boldsymbol{f})$ conditional on the set (vector) $\boldsymbol{f}$ of current financial indices of a borrower is represented in L. Philosophov, V. Philosophov (2002) and L. Philosophov, J. Batten, V. Philosophov (2003).

The methodology is based on enhanced Bayesian multi-alternative models that account for non-normality and interdependence of predictive (explanatory) variables.

The proper choice of financial indices $\boldsymbol{f}$ is of great importance because informative prognostic variables enhance default prediction and increase accuracy of assessing loss distribution function, capital requirement and risk-weighted assets. L. Philosophov, J. Batten, V. Philosophov (2003) proposes two groups of predictive variables.

Group of indices of a firm’s current financial position includes four financial ratios:

- f_1 - Working Capital / Total Assets;
- f_2 - Retained Earnings / Total Assets;
- f_3 - Earnings Before Interest and Tax / Total Assets;
- f_4 - Interest Payments / Total Assets.

Good predictive power of above ratios was confirmed by parts in many studies starting from Altman (1968).

Another group of indices is first proposed in L. Philosophov, J. Batten, V. Philosophov (2003). It is derived from a schedule of paying off a firm’s long-term debt and includes:

- g_1 - portion of long-term debt due within the first year starting from the date of the firm’s last financial statements;
- g_2 - portion of long-term debt due within the second year;
- ………………………………………………………………
- g_m - portion of long-term debt due within the m-th year.

The set of factors $g_1,...g_n$ must comprise all available data including those corresponding to zero payments; variable g_k includes all firm’s long-term payments due in k-th year, not only those from current credit.

Relevant literature studies also variables of market anticipation of bankruptcy of a publicly traded firm. They are not studied in above cited papers but can be easily incorporated in the proposed models.

4. Assessing credit loss distribution for a single borrower. Comparing Bayesian multi-period and Basel II models.

The comparison of the Basel II and BMP models in measuring credit risk of a single borrower is executed along two lines: these are comparison of principal features of the two models and comparative numerical estimations that they provide.

Principal features of the models are represented in the table 1.

Table 1.

MAIN FEATURES	BASEL II MODEL	BMP MODEL
Input data	Probability of default in one year (PD). Maturity time (M). Loss given default (LGD). Exposure at default (EAD).	Last accounting statements of a firm. Schedule of repaying of long-term debt Schedule of planned bank-client cash flows
Probabilities of default	One-year unconditional probability (PD) set in accordance with a firm's rating	Calculated conditional default probabilities for successive years until debt maturity depending on firm's initial financial indices.
Debt maturity	Effective (mean weighted) maturity	True time schedule of cash flows
Loss given default	Preset, non-random	Random, calculated in dependence on non-executed part of cash flow schedule and recovery rate.
Outputs	Marginal loss, risk-weighted assets	Marginal loss, risk-weighted assets

One can see that BMP model much more closely traces economic and financial essence of lending process and account for individual characteristics of a borrower.

Comparison of quantitative outputs of the two models in typical situations is represented in the figure 1.

Curves in the figure represent cumulative credit loss distributions of corporate borrower calculated by means of Basel II model (curve 4) and by BMP model (1,2,3).

A firm borrows U dollars for five years and must pay annual interest at rate 10%.

Basel II credit loss distribution includes expected and unexpected losses and is calculated as dependence between Capital Requirement K (paragraph 241 of the New Basel Capital Accord, version of 2003) and varying confidence probability P_c, which in original Basel II model is set constant 0,999. Numerical data used to calculate this distribution are given in the caption to the figure.

Credit loss distribution (1) is ex-ante distribution calculated by means of BMP model. For better visibility annual default rate is chosen relatively high – 4,5% as it really was in USA in 2001 (see Moody's Investor Service - 2002). The same value was assigned to PD. Curves 2,3 represent ex-post distributions calculated for the same default rate and individual financial indices of firms with good (Lechters Inc., 1993) and bad (Imperial Sugar Co., 2000) financial positions.

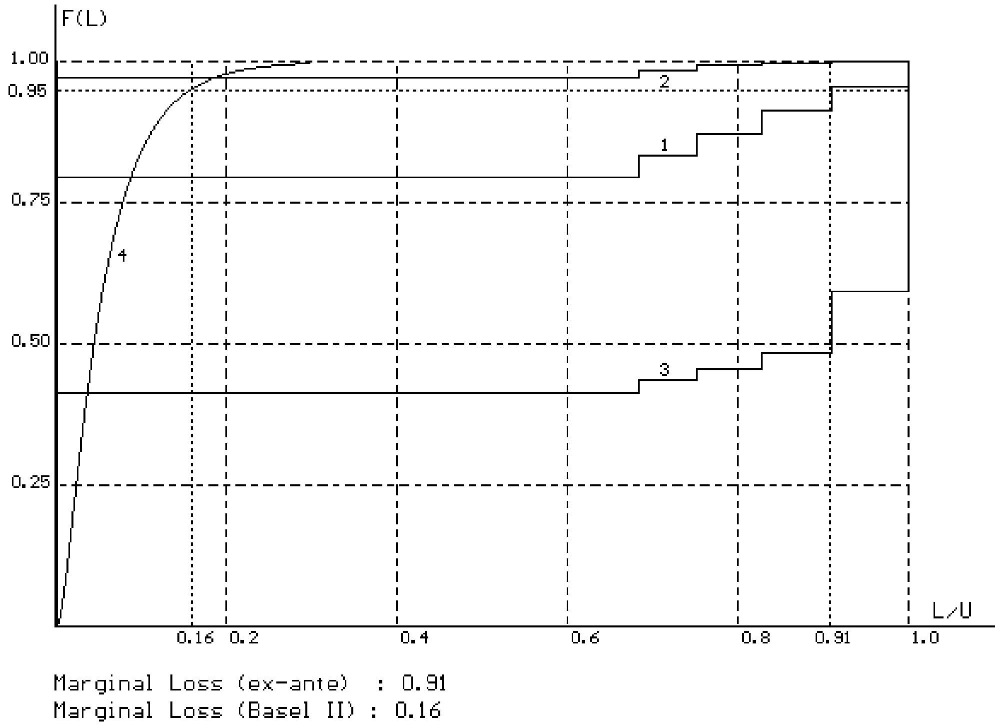

Figure 1. Cumulative loss distribution functions for single corporate borrowers.
BMP model – stepwise curves 1,2,3:

- 1 – ex-ante distribution with annual default rate 4,5%
- 2 – ex-post distribution for Lechters Inc., 1993.
- 3 – ex-post distribution for Imperial Sugar Co., 2000.

Basel II model – continuous curve (4).
M = 5; PD = 4,5% - true USA default rate in 2001; LGD = 1,0; R = 0,133 - as calculated by Basel II model.

One can see that losses assessed by the two models greatly differ from each other; the only resemblance is that all them vary in range $0 \div 1$.

All BMP distributions admit zero losses that correspond situations when a default does not occur. Basel II distribution does not admit zero losses.

Horizontal dotted line in the figure corresponds to confidence level $P_c = 0{,}95$, which is chosen relatively low for better visibility. Abscissa of its intersect with the graph of cumulative loss distribution determines marginal loss. One can see that marginal loss is $L_{m1} = 0{,}91$ in case of ex-ante distribution (curve 1), $L_{m2} = 0{,}0$ in case of Lechters Inc. (curve 2, in this case probability of zero losses is more than confidence probability),. $L_{m3} = 1{,}0$ in case of

Imperial Sugar Co. (curve 3). Basel II model does not capture these situations. Marginal loss corresponding to Basel II distribution is $L_{m4} = 0{,}16$ in all above cases.

The data of the figure 1 evidence also that accounting for the individual financial position of a (corporate) borrower is very important.

In attempt to reveal sources of discrepancy between the two models I checked the Wehrspohn's inference of Basel II formulae. Original Basel II inference is unavailable, but the doubtful term is identical in Basel II model and in Wehrspohn (2003). Given very specific form of this term one can be sure that inferences are highly correlated.

It was found that formula for capital requirement K in Basel II item 241 (The New Basel Capital Accord, ver. 2003) is erroneous (detailed analysis one can find in Appendix).

Erroneous formula in Basel II notation is

$$K = LGD \times N[(1-R) \wedge -0.5 \times G(PD) + (R/(1-R) \wedge 0.5 \times G(0.999)];$$

in more usual notation of Wehrspohn (2003) it is

$$K = LGD \times \Phi\left(\frac{\Phi^{-1}(p) + \sqrt{\rho} \cdot \Phi^{-1}(0{,}999)}{\sqrt{1-\rho}}\right),$$

where $\Phi(\bullet)$ is cumulative standard normal distribution function and $\Phi^{-1}(\bullet)$ inverse function to $\Phi(\bullet)$; $p = PD$, ρ is the same as R.

In both formulae 0,999 is specific value of confidence probability P_c. Some Basel II formulae for capital requirement contain additional maturity adjustment term.

Correct expression for K in Basel II model (for an individual borrower) is much more simple:

$K = 0$ if $PD < 1 - P_c$ and

$K = LGD$ if $PD > 1 - P_c$.

Note that after corrections correlation factor R disappeared from Basel II formulae for required capital and risk-weighted assets. This is quite natural for a single borrower.

In some formulae for Capital requirement K Basel II model uses additional multiplier

$$T = (1 - 1.5 \times b(PD)) \wedge -1 \times (1 + (M - 2.5) \times b(PD)),$$

whose destination is maturity adjustment. In the current examples its influence is insignificant.

The above one and other examples evidence that in assessing credit risk of a single borrower Basel II model is very imprecise and highly undervalues credit risks.

To improve the situation radically one can use BMP model instead of the Basel II model.

5. Portfolio of one-period credits.

Consider now portfolio of one-period credits. A bank lends to each borrower U dollars at time t_0 and at the end of credit period receives the loan back with extra interest payment. Time moment t_0 can vary for different clients in portfolio.

This case corresponds to $M = 1$ in above formulae, and from (11, 12) one can find $P_D(T_1) = \pi$ and $P_D(T_{1+}) = 1 - \pi$.

Cumulative loss distribution (as follows from (7, 8) is

$$F(L) = (1 - \pi) \cdot 1(L) + \pi \cdot 1(L - U \cdot (1 - \beta)),$$

and cumulative distribution of LGD is

$$F(LGD) = 1(LGD - (1 - \beta)).$$

The last expression means that $LGD = 1 - \beta$ with probability 1.

Within active credit period each of n borrowers in portfolio can either default with probability π or stay solvent with probability $1 - \pi$. Each default brings to a bank loss $U \cdot (1 - \beta)$. If default events for different clients are independent, portfolio status at the end of credit period is equivalent to one of possible issues of n Bernoulli tests.

Probability of exactly k defaults in portfolio is equal to

$$P(k) = \frac{n!}{k!(n-k)!} \pi^k (1 - \pi)^{n-k}, \tag{13}$$

and bank's loss in this case is equal to:

$$L = k \cdot (1 - \beta) \cdot U. \tag{14}$$

Taking in account (13, 14) cumulative loss distribution of the portfolio can be determined as:

$$F(L) = \sum_{k=0}^{n} P(k) \cdot 1(L - k \cdot (1 - \beta) \cdot U), \tag{15}$$

where again function $1(x) = 0$ if $x < 0$ and $1(x) = 1$ if $x \geq 0$.

To bring expressions (13, 15) to conditions of Basel II, one must also take in account correlation. Basel (1999) document explains effect of correlation by (common for all borrowers) influence of macroeconomic environment that changes their propensity to default. For portfolio BMP (PBMP) model this means that default rate π must be considered random. Basel II inference proposes for π the following model (see Wehrspohn (2003) and Appendix to this paper):

$$\pi = \pi(Y) = \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{R} \cdot Y}{\sqrt{1-R}}\right), \tag{16}$$

where PD is interpreted as probability of default within one year, R – correlation, Y - random variable with standard normal distribution $N(Y)$, Φ - function of cumulative standard normal distribution, Φ^{-1} - inverse function of Φ.

As the result cumulative loss distribution of portfolio of one-period credits is determined by the following formula:

$$F(L) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \int_{-\infty}^{\infty} \pi(Y)^k (1-\pi(Y))^{n-k} \cdot N(Y) \cdot dY \cdot 1(L - k \cdot (1-\beta) \cdot U). \tag{17}$$

Taking some care with factorials (which are very large numbers), one can use (17) to calculate exact loss distribution of portfolio of n one-period credits. This distribution is represented in figure 2 (stepwise curve) for portfolio of 100 borrowers and $\beta = 0$ (and hence $LGD = 1)$ - no recoveries.

Continuous curve represents cumulative loss distribution determined by Basel II model. This again is dependence between Capital Requirement K and varying confidence probability P_c, which in original Basel II model is set as constant $0,999$. In this model we also have set $LGD = 1$.

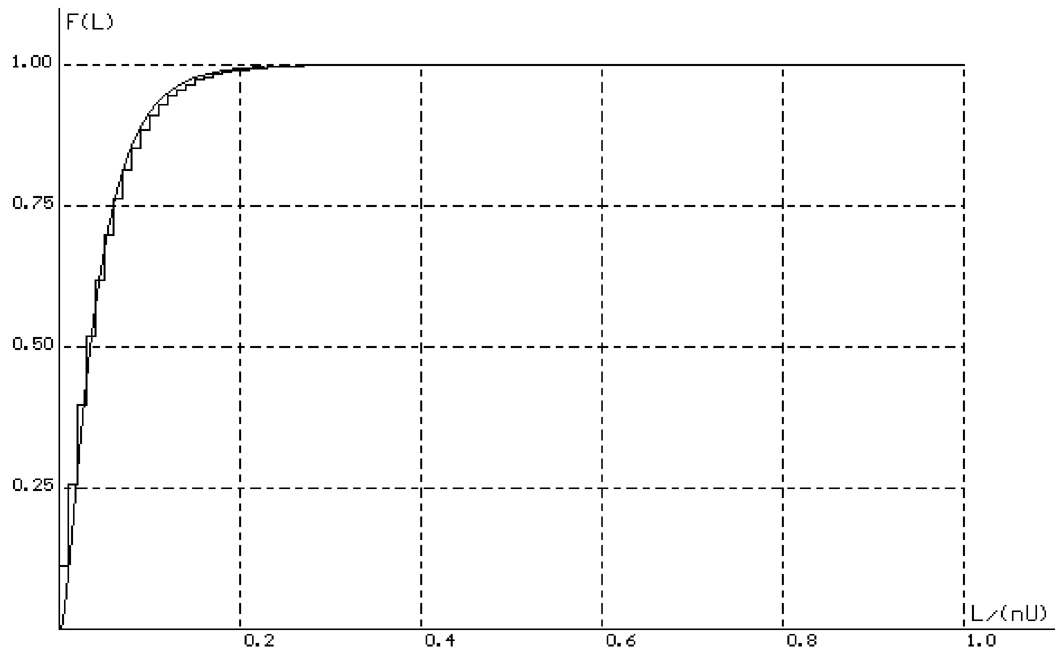

Figure 2. Cumulative loss distribution functions for portfolio of 100 borrowers. Loss is calculated as the share of total portfolio exposure:

Exact (Portfolio BMP) model – stepwise curve, Basel II model – continuous curve.

M = 1; n = 100; PD = 4,5% - true USA default rate in 2001; LGD = 1,0; R = 0,133 - as calculated by Basel II model.

For parameter values given in the caption to the figure, marginal loss, corresponding to the confidence probability $P_c = 0{,}999$, is assessed by the exact model as *0,28* of total portfolio exposure (28 borrowers of 100 in portfolio can default simultaneously with probability more than $1 - P_c = 0{,}001$), while Basel II model assesses this loss as *0,27* of total exposure; accuracy of Basel II is 1%.

One can see that (in contrast with the previous section) for portfolio of 100 clients conformity between exact (Portfolio BMP) and Basel II models is rather good.

Using exact formulae (16, 17) one can assess how portfolio size influences accuracy of Basel II model.

In figure 3 exact cumulative loss distribution and those determined by Basel II model are calculated for portfolio of 30 borrowers.

Exact model assesses that loss will be zero with probability *0,37*. This is probability that no one of 30 borrowers will default while Basel II model does not admit zero losses.

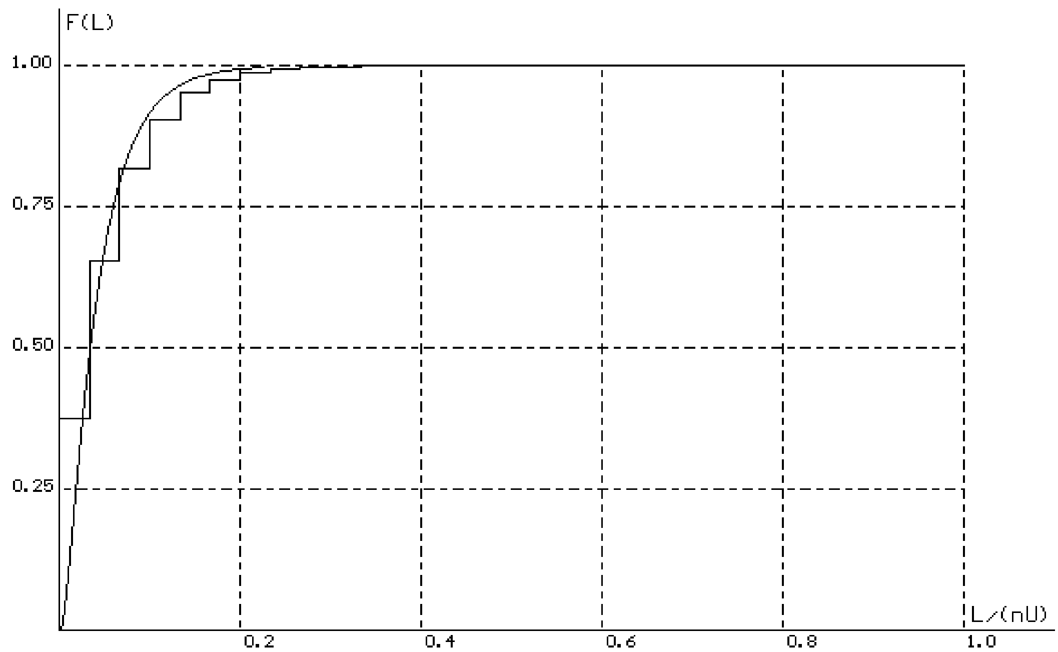

Figure 3. Cumulative loss distribution functions for portfolio of 30 borrowers. Loss is calculated as the share of total portfolio exposure:

PBMP model – stepwise curve, Basel II model – continuous curve.
M = 1; n = 30; PD = 4,5%; LGD = 1,0; R = 0,133 (as calculated by Basel II).

For parameter values given in the caption to the figure marginal loss, corresponding to confidence probability $P_c = 0{,}999$, is assessed by the exact model as *0,3* of total portfolio exposure (9 borrowers of 30 in portfolio can default simultaneously with probability more than $1 - P_c$), while Basel II model again assesses this loss as *0,27* of total exposure; accuracy of Basel II is 10%.

In figure 4 the same probability distributions are calculated for portfolio of 10 borrowers. In this case distinctions between exact and Basel II distributions are still more significant.

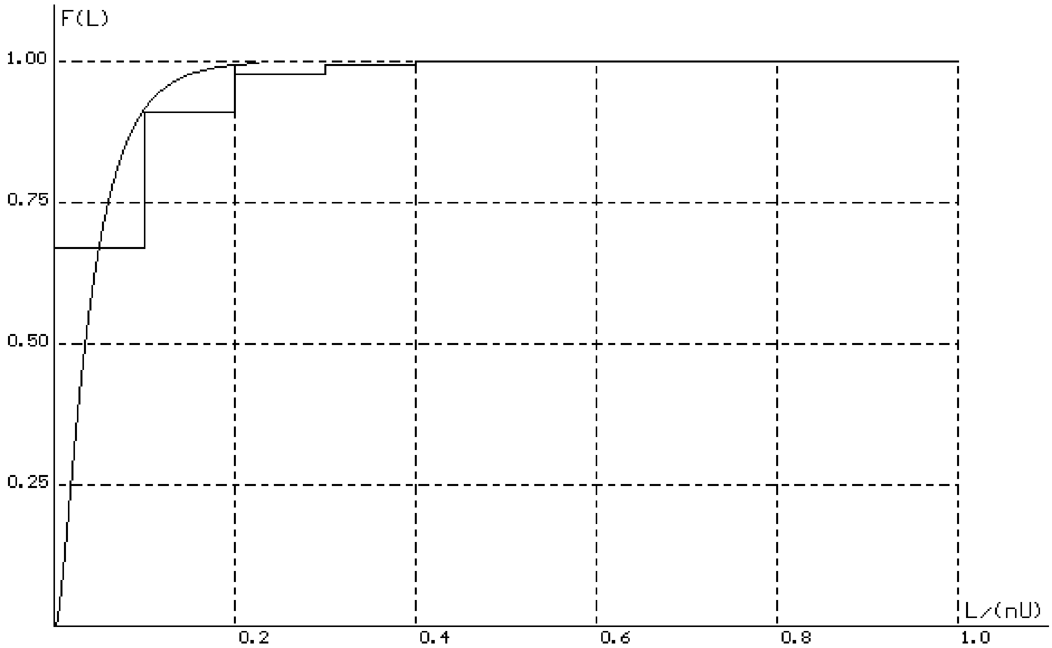

Figure 4. Cumulative loss distribution for portfolio of 10 borrowers. Loss is calculated as the share of total portfolio exposure:
Exact model – stepwise curve, Basel II model – continuous curve.
M = 1; n = 10; PD = 4,5%; LGD = 1,0; R = 0,133.

Marginal loss, corresponding to confidence probability $P_c = 0{,}999$, is assessed by the exact model as *0,4* of total portfolio exposure (4 borrowers of 10 can default simultaneously with probability more than $1 - P_c$) while assessment of Basel II does not change - *0.27* of total exposure (accuracy of Basel II is 32,5%).

Finally the same distributions can be calculated for a single borrower. Formula (17) for cumulative loss distribution is this case reduces to

$$F(L) = (1-\bar{\pi}) \cdot 1(L) + \bar{\pi} \cdot 1(L-U), \tag{18}$$

where $\bar{\pi} = \int_{-\infty}^{\infty} \pi(Y) \cdot N(Y) \cdot dY$ - mean value of π. One can calculate that $\bar{\pi} = PD.$

The resulting cumulative distribution (18) is represented in figure 5 together with Basel II distribution which is the same curve as before. . Exact distribution has only one step that corresponds to cumulative probability $F(L) = 1 - PD.$ Marginal loss in exact model is assessed as $L_{0,999} = U$ - credit can be fully lost with probability more than $1 - P_c$, while Basel II again assesses marginal loss as $0,27$ which is obviously incorrect.

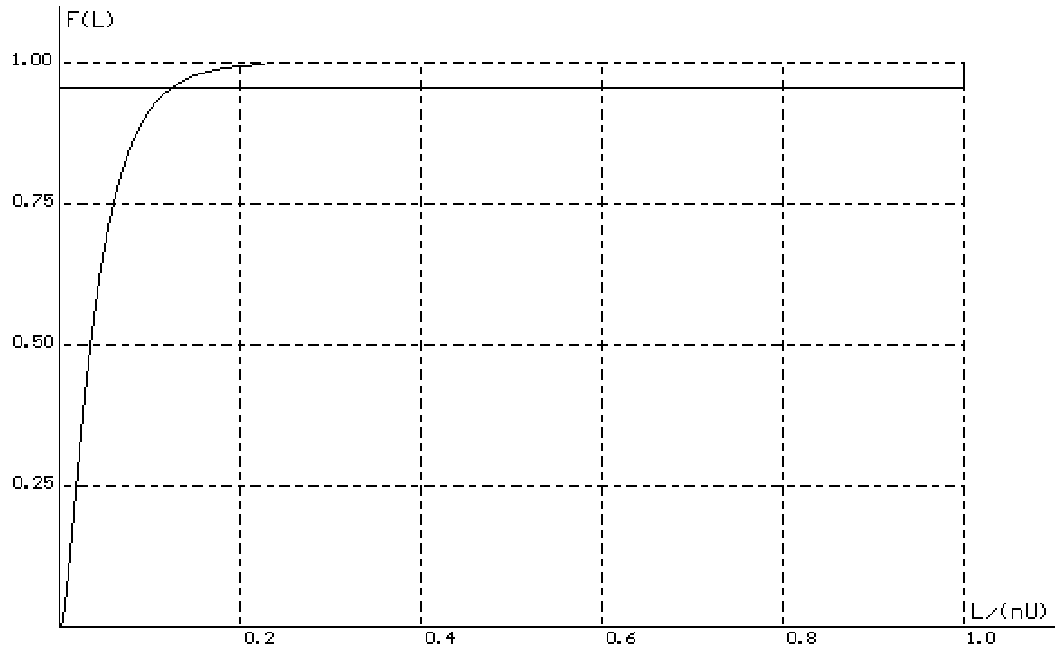

Figure 5. Cumulative loss distribution for a single borrower.
Exact model – stepwise curve, Basel II model – continuous curve.
M = 1; n = 1; PD = 4,5%; LGD = 1,0; R = 0,133.

The model, described by exact expressions (16,17), (taking in account its good coincidence with Basel II when portfolio is large) presents alternative look on concepts of Probability of Default (PD) and Correlation (R) considered within Basel II approach.

In exact model PD is unobservable parameter, while observable is current default rate π. As we noted earlier PD is mean value of π. Hence time base used for determination of PD must be very large, given relatively slow alteration of default rates.

One must also note that distribution of default rate π, used in Basel II model, is rather specific. Cumulative distribution as determined by (16) is:

$$F(\pi) = 1 - \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{1-R}\cdot\Phi^{-1}(\pi)}{\sqrt{R}}\right).$$

It is based on very indirect considerations and ignores rich statistical data concerning default rates.

On the other hand using current default rates π instead of average rate PD is preferable because it increases accuracy of assessing current credit risk. Hence *accounting for correlation in Basel II model is unnecessary.* In this case formulae (13, 14, 15) can be used to exactly assess credit loss distributions.

6. Portfolio of multi-period credits.

In the multi-period case credit process for each borrower has $M+1$ possible issues, each of them with its specific probability $P_D(T_m)$ and lost amount L_m, as described in the section 2. For portfolio of n multi-period credits one must consider polynomial (instead of binomial) tests (Feller 1966). In common case one can find in portfolio k_1 defaults within the first year, … k_M defaults within the M-th year, k_{M+1} borrowers avoid default within active credit period $(k_1 + ... + k_M + k_{M+1} = n)$. Probability of such issue is polynomial probability

$$P(k_1,...k_M,k_{M+1}) = \frac{n!}{k_1!\cdot ...\cdot k_M!\cdot k_{M+1}!}\cdot P_D^{k_1}(T_1)\cdot ...\cdot P_D^{k_M}(T_M)\cdot P_D^{k_{M+1}}(T_{M+1}), \tag{19}$$

and total portfolio loss is

$$L_p(k_1,...k_M,k_{M+1}) = k_1\cdot L_1 + ... + k_M\cdot L_M + k_{M+1}\cdot L_{M+1}. \tag{20}$$

Note that in accordance with formulae (9,10) probabilities $P_D(T_m)$ depend on annual default rates $P_D(T_m) = P_D(T_m/\pi_1,...\pi_m)$. To take them maximally near to Basel II we consider π_m random but the same during all credit period $\pi_m \equiv \pi$. Probabilistic properties of default rate π are again determined by formula (16).

As the result we obtain following expression for cumulative loss distribution:

$$F(L) = \sum_{k_1}...\sum_{k_M}\frac{n!}{k_1!...k_{M+1}!}\cdot\int P_D^{k_1}(T_1/\pi(Y))\cdot ...\cdot P_D^{k_{M+1}}(T_{M+1}/\pi(Y))\cdot P(Y)\cdot dY\cdot 1(L-L_p)$$

, (21)

where k_i take all possible values that satisfy the restriction $(k_1 + ... + k_M + k_{M+1} = n)$ and L_p must be taken from (20).

Expression (21) is rather complex and bulky, but rational organization of computations allows for its quick calculation.

In figure 6 we represent cumulative loss distributions calculated by means of exact (PBMP) model (1) and Basel II model (2) for portfolio of 100 three-year credits with annual interest payments. Remember that for one-year credits both models reveal good conformity of assessed credit loss distributions in such a large portfolio.

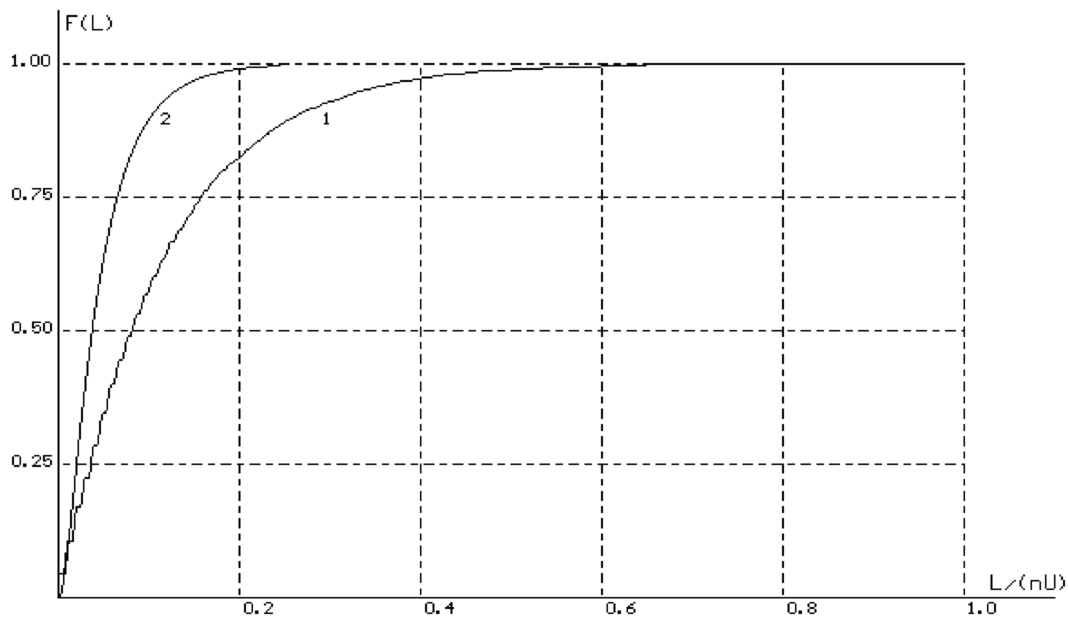

Figure 6. Cumulative distributions of credit losses for portfolio of three-year credits according to exact (1) and Basel II (2) models.

M = 3; n = 100; PD = 4,5%; $\beta = 0$ - no recoveries; mean LGD = 0,938 (as calculated by PBMP model); R=0,133 (as calculated by Basel II model).

In case of three-year credits Basel II model undervalues credit losses; maturity adjustment of the model is insufficient. Note that in PBMP model LGD is calculable value; the same value was used in Basel II model.

In the figure 7 the same curves are represented for portfolio of five-year credits. One can see that in this case Basel II model undervalues credit losses still more significantly.

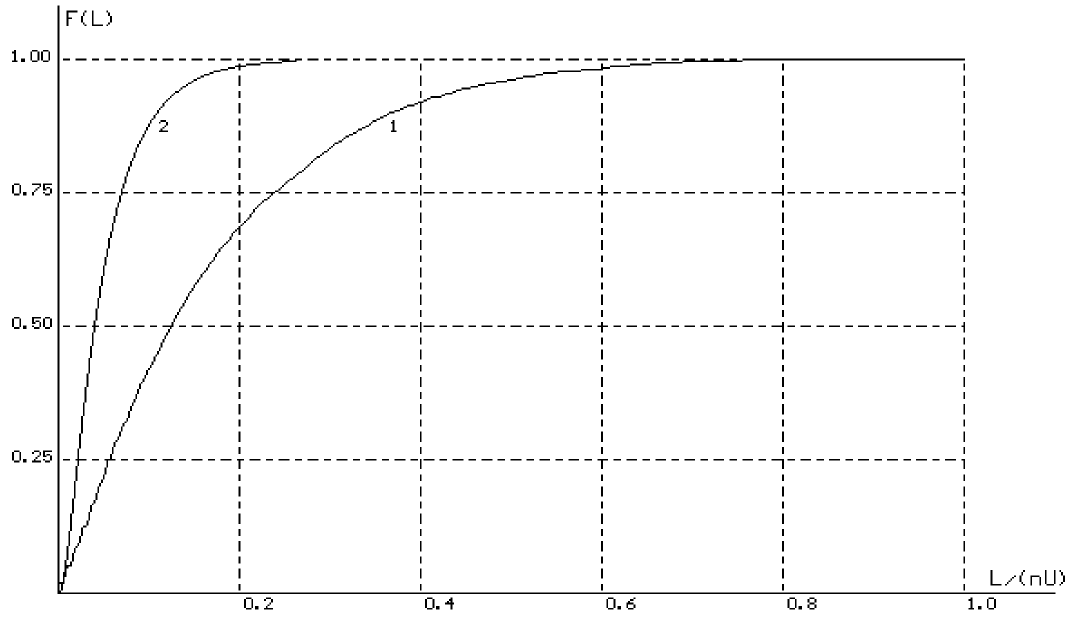

Figure 7. Cumulative distributions of credit losses for portfolio of five-year credits according to exact (1) and Basel II (2) models.

M = 5; n = 100; PD = 4,5%; $\beta = 0$ - no recoveries; mean LGD = 0,883 (as calculated by PBMP model); R=0,133 (calculated).

7. Conclusion and propositions.

The above calculations and graphs evidence that area of correct assessments of the Basel II credit risk model is restricted by one-period credits and medium or large portfolios of about 30 or more borrowers. For portfolios of lesser size down to individual borrowers stepwise character of true credit loss distribution is poorly approximated by continuous Basel II curves.

For single corporate borrowers their individual financial characteristics drastically influence their credit risks.

For multi-period credits Basel II model undervalues credit losses the more the more periods (intermediate payments) covers active credit interval. Maturity adjustment in the Basel II model appears to be insufficient (if at all possible).

The study proves that accounting for correlation in assessing risk of individual borrowers and credit portfolios is unnecessary. Correlation is caused by changes in default rate (common for all borrowers) in dependence on macroeconomic situation. Current default rate π can be measured and used instead of mean default rate PD in credit risk estimations.

Exact BMP and PBMP models are more complex than Basel II model, but nevertheless they are easily available for PC calculations that take no more than one minute for one distribution. Such models can be used in bank practice and flexibly account for peculiarities of specific portfolios.

The models can be implemented as a standardized software programs, developed under the auspices of Basel Committee and distributed among relevant banks.

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Appendix.

Basel Committee on Banking Supervision (2004) document "International Convergence of Capital Measurement and Capital Standards" in its item 272 presents formula for calculation of required capital K, which in this document is equal to "unexpected losses". To receive full (marginal) losses one can add "expected" (mean) losses. The formula for full losses was presented in the preceding version of Basel II (The New Basel Capital Accord (2003), item 241). This formula is:

$$K = LGD \times N[(1-R) \wedge -0.5 \times G(PD) + (R/(1-R) \wedge 0.5 \times G(0.999)],$$

and in more usual notation and suitable for perception form can be rewritten as

$$K = LGD \times \Phi\left(\frac{\Phi^{-1}(p) + \sqrt{\rho} \cdot \Phi^{-1}(0{,}999)}{\sqrt{1-\rho}}\right), \tag{A.1}$$

where $\Phi(\bullet)$ is cumulative standard normal distribution function and $\Phi^{-1}(\bullet)$ inverse function to $\Phi(\bullet)$; $p = PD$, ρ is the same as R.

Formula of this type was inferred in Wehrspohn (2003) for marginal credit loss (percentile) in portfolio of defaulting clients.

Given specific form of this expression quite probable supposition is that Basel II has inferred it in the same underlying assumptions.

The following comment proves that the formula (A.1) for K is invalid for a single client and presents the improved formulae.

Wehrspohn's inference relates to portfolio of clients; in the following text his formulae are reduced to a single client as it is done in the Basel II.

In his paper Wehrspohn assigns to a client (firm) a random risk index X, which has standard normal distribution $N(0,1)$ with zero mean and variance equal to 1. A firm defaults if X falls below threshold α; probability of this event is known value p (or PD in Basel II notation) and hence $\alpha = \Phi^{-1}(p)$, where again $\Phi^{-1}(\bullet)$ denotes inverse of cumulative normal distribution.

Further he decomposes X into two additive parts Y, Z:

$$X = \sqrt{\rho} \cdot Y + \sqrt{1-\rho} \cdot Z,$$

where Y represents systematic risk and Z - idiosyncratic risk. Both Y and Z have the same standard normal distribution $N(0,1)$.

If a borrower defaults a bank bears loss (LGD) denoted in Wehrspohn's paper as λ, which is non-random predetermined value.

Next Wehrspohn freezes Y and founds that probability of client default conditional on Y is

$$P(client\ defaults \mid Y) = \Phi(\frac{\Phi^{-1}(p)-\sqrt{\rho}\cdot Y}{\sqrt{1-\rho}}).$$

Then in course of proving Theorem 1 he states that loss given Y is determined by the expression:

$$\text{Loss} \mid \text{Y} = \lambda\cdot E\cdot\Phi(\frac{\Phi^{-1}(p)-\sqrt{\rho}\cdot Y}{\sqrt{1-\rho}}),\ (E = EAD). \tag{A.2}$$

In this place we must note this is mean loss. In this point of inference loss given Y is the random variable, which in application to a single client takes only two possible values: loss is $L = \lambda\cdot E$ (client defaults) with probability π

$$\pi = \Phi(\frac{\Phi^{-1}(p)-\sqrt{\rho}\cdot Y}{\sqrt{1-\rho}}) \tag{A.3}$$

and $L = 0$ (client does not default) with probability $1-\pi$. Mean loss is

$$Mean\ Loss \mid Y = 0\cdot(1-\pi)+\lambda\cdot E\cdot\pi = \lambda\cdot E\cdot\pi$$

This is just the same as formula (A.2).

Further Wehrspohn considers mean loss given Y from (A.2) to be random because of randomness of Y and calculates percentiles of resulting distribution of Mean Loss, which can be correct only for large portfolios, not for a single client.

In fact, to determine marginal loss one must calculate percentiles of unconditional loss distribution density, which is determined as:

$$P(L) = \int P(L \mid Y)\cdot P(Y)\cdot dY. \tag{A.4}$$

In this formula $P(L \mid Y)$ is probability density (not mean value) of Loss given Y - above described random variable with two possible values. One can write $P(L \mid Y)$ in the following form:

$$P(L \mid Y) = (1-\pi)\delta(L)+\pi\cdot\delta(L-\lambda E)$$

where $\delta(\bullet)$ is well known in mathematics symbolic delta-function, which may be thought of as a normal probability density with very small (almost zero) variance. $\delta(L)$ is concentrated in very narrow area around $L = 0$, while $\delta(L-\lambda E)$ is concentrated around $L = \lambda E$.

As the result of averaging in (A.4) we obtain

$$P(L) = (1-\bar{\pi})\delta(L)+\bar{\pi}\cdot\delta(L-\lambda E) \tag{A.5}$$

where $\bar{\pi}$ is mean value of π (π is dependent on Y).

$$\bar{\pi} = \int \pi(Y) \cdot P(Y) \cdot dY . \tag{A.6}$$

The later integral can be calculated analytically.

The result is very interesting and quite predictable: $\bar{\pi} = PD$ - probability of default.

This result one could receive directly considering distribution of X without its decomposition into Y, Z ; the decomposition has little sense for a single client.

For practical needs of the New Capital Accord is important to note that unconditional distribution $P(L)$ determined by (A.5) stays to be discrete with only two possible values $L = 0$ and $L = \lambda E$. The 99,9%-quantile (99,9-percentile) of L is $L_{99,9\%} = 0$ if $PD < 1 - 0,999 = 0,001$ and $L_{99,9\%} = \lambda E$ if $PD > 0,001$.

Capital requirement K in Basel II formulae (items 241, 299, 301) becomes trivial:

$K = 0$ if $PD < 0.001$ and

$K = LGD$ if $PD > 0.001$.

These expressions could be more correct if PD corresponds to probability of a client's default within M years - period until debt maturity (not within one year as it is established at present time).